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since

$$(1-a)(1-b) = (1-b)(1-a) \pmod{M^3}.$$

We know that  $(1-a)^2, (1-b)^2, (1-a)(1-b)$  forms a  $K$ -basis of  $M^2/M^3$ . Hence  $u \cdot v \neq 0$  and  $v \cdot u \neq 0 \pmod{M^3}$ . Otherwise

$$x_1x_2 = y_1y_2 = x_1y_2 + x_2y_1 = 0$$

and  $x_1y_2 - x_2y_1 = 0$  contrary to the fact that  $\{u, v\}$  gives a basis of  $M/M^2$ .

Thus  $uv, vu \in B$  satisfy  $uv = vu \pmod{M^3}$  and  $uv \neq 0, vu \neq 0 \pmod{M^3}$ .

It follows that  $uv = vu$ . In fact more generally, if  $u_1, u_2 \in B \setminus M^k$  and  $u_1 = u_2 \pmod{M^k}$  then  $u_1 = u_2$ . Proof:  $B \cap M^k$  is a basis of  $M^k$ , thus  $u_1 - u_2 = \sum_{u \in B \cap M^k} \lambda_u u$ . This is possible only if  $u_1 - u_2 = 0$ .

### § 3. THE GROUP OF QUATERNION UNITS

Let  $Q$  be defined by generators and relations:

$$Q = \langle a, b : a^4 = 1, b^2 = a^2, ab = b^3a \rangle.$$

Set  $i = a, j = b, k = ab$  and  $c = a^2$ . Then

$$Q = \{1, c, i, ci, j, cj, k, ck\}.$$

**PROPOSITION 2.** *Let  $K$  be a field of characteristic 2. The group algebra  $K[Q]$  possesses a filtered multiplicative basis if and only if  $K$  contains a primitive cube root of unity.*

*Proof.* If  $K$  contains a primitive cube root of unity, say  $\omega$ ,  
let

$$B = \{1, u, v, uv, vu, u^2, v^2, u^3\},$$

where

$$\begin{aligned} u &= \omega i + \omega^2 j + k \\ v &= \omega^2 i + \omega j + k. \end{aligned}$$

It is easily verified that  $B$  is a filtered multiplicative basis.

Conversely, suppose that  $K[Q]$  possesses a filtered multiplicative basis  $B$ .

Observe that  $\{1+i, 1+j\}$  is a  $K$ -basis of  $M/M^2$ , where again  $M = \text{rad } K[Q]$ . Also  $\{1+c, 1+i+j+k\}$  is a  $K$ -basis of  $M^2/M^3$ . Since  $B \cap (M/M^2) = \{u, v\}$  must be a  $K$ -basis of  $M \text{ mod } M^2$ , we have

$$u = x_1(1+i) + y_1(1+j) \text{ mod } M^2,$$

$$v = x_2(1+i) + y_2(1+j) \text{ mod } M^2,$$

with  $x_1, y_1, x_2, y_2 \in K$  and  $x_1y_2 + x_2y_1 \neq 0$ .

Now

$$\begin{aligned} u \cdot v &= (x_1x_2 + y_1y_2 + x_2y_1)(1+c) \\ &\quad + (x_1y_2 + x_2y_1)(1+i+j+k) \text{ mod } M^3, \end{aligned}$$

$$\begin{aligned} v \cdot u &= (x_1x_2 + y_1y_2 + x_1y_2)(1+c) \\ &\quad + (x_1y_2 + x_2y_1)(1+i+j+k) \text{ mod } M^3. \end{aligned}$$

Therefore,

$$u \cdot v + v \cdot u = (x_1y_2 + x_2y_1)(1+c) \text{ mod } M^3 \neq 0 \text{ mod } M^3,$$

and so  $u \cdot v \neq v \cdot u$ .

We must have  $uv \in B \cap (M^2 \setminus M^3)$  since the  $(1+i+j+k)$ -coordinate of  $u \cdot v$  is non-zero. Similarly  $v \cdot u \in B \cap (M^2 \setminus M^3)$ . But  $\dim(M^2/M^3) = 2$  and so

$$B \cap (M^2 \setminus M^3) = \{uv, vu\}.$$

Consider the element  $u^2 \in M^2$ . Either  $u^2 = uv$  or  $u^2 = vu$  or  $u^2 \in M^3$ . But  $u^2 = (x_1^2 + y_1^2 + x_1y_1)(1+c) \text{ mod } M^3$ .

Since the  $(1+i+j+k)$ -coordinate of  $u^2$  is 0, we have  $u^2 \neq uv$ ,  $u^2 \neq vu$ .

Hence  $u^2 \in M^3$ . This implies  $u^2 = 0$ , and it follows that the quadratic form  $x_1^2 + y_1^2 + x_1y_1$  represents 0 non-trivially in  $K$  and  $w = y_1/x_1$  is a primitive cube root of unity in  $K$ .