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Artikel: SOME EXAMPLES OF GROUP ALGEBRAS WITHOUT FILTRED MULTIPLICATIVE BASIS

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§ 1. Preliminary remarks

Note that if R is a finite dimensional K-algebra and B is a filtred multiplicative basis (as defined above), then $B \cap \{\text{rad}(R)\}^n$ is a K-basis of $\{\text{rad}(R)\}^n$ for all $n \geqslant 1$. Indeed, the set $B \cap \{\text{rad}(R)\}^n$ is linearly free over K since B is. By hypothesis, $B \cap \text{rad}(R)$ is a K-basis of $\text{rad}(R)$ and thus the set of products $b_1 \cdot ... \cdot b_n$ with $b_i \in B \cap \text{rad}(R)$ is a generator system for $\{\text{rad}(R)\}^n$. But all such products $b_1 \cdot ... \cdot b_n$ are either 0 or belong to $B \cap \{\text{rad}(R)\}^n$. Hence, $B \cap \{\text{rad}(R)\}^n$ generates $\{\text{rad}(R)\}^n$ over K.

The case of a finite abelian group G is easy to understand:
Let

$$G_p = \langle a_1 \rangle \times \langle a_2 \rangle \times ... \times \langle a_r \rangle$$

be a decomposition of the p-Sylow subgroup G_p of G as a direct product of cyclic groups of orders $p^{n_1}, ..., p^{n_r}$ respectively. Let $G = G_p \times H$, where $| H |$ is prime to p. Then,

$$B = \{(a_1 - 1)^{m_1} \cdot (a_2 - 1)^{m_2} ... (a_r - 1)^{m_r} \cdot h \mid 0 \leqslant m_i \leqslant n_i - 1, h \in H\}$$

is a filtred multiplicative basis of KG for any field K of characteristic p.

If we insist that the elements of B outside $\text{rad}(R)$ should be orthogonal idempotents, then we have to require that K be algebraically closed, as otherwise KH itself need not have a filtred multiplicative basis B satisfying

MB3) If $e, e' \in B \backslash \text{rad}(R)$, $e \neq e'$, then $e^2 = e$ and $e \cdot e' = 0$.

Observe that, more generally, if B_1, B_2 are filtred multiplicative bases for KG_1 and KG_2, then $B_1 \times B_2$ is a filtred multiplicative basis for $K[G_1 \times G_2]$.

In the next paragraphs we will examine examples of p-groups.

If G is a p-group, and K a field of characteristic p, then $\text{rad}(KG)$ is the augmentation ideal

$$\text{rad}(KG) = \{\sum_{g \in G} \alpha_g g \mid \sum_{g \in G} \alpha_g = 0\}$$

Note also that in that case, a filtred multiplicative basis B for KG necessarily contains 1. Indeed, $\dim_k \{KG/\text{rad}(KG)\} = 1$. If $e \in B$ is the unique element outside $\text{rad}(KG)$, then $e^2 \notin \text{rad}(KG)$ and therefore $e^2 = e$. But KG is local, thus $e = 1$. (Alternatively, $e = 1 + r$ with $r \in \text{rad}(KG)$ and $e = e^{p^N} = 1 + r^{p^N} = 1$.)

Thus for p-groups, axiom MB3) is automatically satisfied.