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## SOME EXAMPLES OF GROUP ALGEBRAS WITHOUT FILTRED MULTIPLICATIVE BASIS

by Luis PARIS

Let  $R$  be a finite dimensional algebra over a field  $K$ . A *filtred multiplicative basis*  $B$  for  $R$  is a  $K$ -basis of  $R$  such that

MB1)  $b, b' \in B$  implies  $b \cdot b' \in B$  or  $b \cdot b' = 0$ ,

MB2)  $B \cap \text{rad}(R)$  is a  $K$ -basis of  $\text{rad}(R)$ , where  $\text{rad}(R)$  is the radical of  $R$ .

In [B.-G.-R.-S.], R. Bautista, P. Gabriel, A. V. Roiter and L. Salmereon prove that if  $R$  is of finite representation type, i.e. if there are only finitely many isomorphism classes of indecomposable  $R$ -modules, then  $R$  possesses a filtred multiplicative basis. In their introduction, they state: "It is not known to us whether general group-algebras do [have filtred multiplicative bases]".

Of course, it is well known that if  $K$  is a field of characteristic  $p$ , then the group algebra  $KG$  of the finite group  $G$  is of finite representation type if and only if the  $p$ -Sylow subgroups of  $G$  are cyclic. (See e.g. [C.-R.], page 431.)

However, if  $G = C_p \times C_p$ , the direct product of 2 copies of a cyclic group of order  $p$ , generated by  $a$  and  $b$ , then the set

$$B = \{(a-1)^i (b-1)^j \mid 0 \leq i, j \leq p-1\}$$

is a filtred multiplicative basis of  $KG$  for any field  $K$  of characteristic  $p$ , although the representation type of  $K[C_p \times C_p]$  is infinite.

In this note we produce some less obvious examples showing that the group algebras of  $p$ -groups over an algebraically closed field of characteristic  $p$  do not necessarily admit a filtred multiplicative basis.

We also show that for the group of quaternion units  $Q$ , of order 8, and  $K$  of characteristic 2, the algebra  $KQ$  admits a filtred multiplicative basis if and only if  $K$  contains a primitive 3-rd root of unity.

This note is a condensed version of my "Travail de Diplôme" at the University of Geneva. I am grateful to Claude Cibils and Michel Kervaire for valuable suggestions and their encouragements during my work.

## § 1. PRELIMINARY REMARKS

Note that if  $R$  is a finite dimensional  $K$ -algebra and  $B$  is a filtered multiplicative basis (as defined above), then  $B \cap \{\text{rad}(R)\}^n$  is a  $K$ -basis of  $\{\text{rad}(R)\}^n$  for all  $n \geq 1$ . Indeed, the set  $B \cap \{\text{rad}(R)\}^n$  is linearly free over  $K$  since  $B$  is. By hypothesis,  $B \cap \text{rad}(R)$  is a  $K$ -basis of  $\text{rad}(R)$  and thus the set of products  $b_1 \cdot \dots \cdot b_n$  with  $b_i \in B \cap \text{rad}(R)$  is a generator system for  $\{\text{rad}(R)\}^n$ . But all such products  $b_1 \cdot \dots \cdot b_n$  are either 0 or belong to  $B \cap \{\text{rad}(R)\}^n$ . Hence,  $B \cap \{\text{rad}(R)\}^n$  generates  $\{\text{rad}(R)\}^n$  over  $K$ .

The case of a finite abelian group  $G$  is easy to understand:

Let

$$G_p = \langle a_1 \rangle \times \langle a_2 \rangle \times \dots \times \langle a_r \rangle$$

be a decomposition of the  $p$ -Sylow subgroup  $G_p$  of  $G$  as a direct product of cyclic groups of orders  $p^{n_1}, \dots, p^{n_r}$  respectively. Let  $G = G_p \times H$ , where  $|H|$  is prime to  $p$ . Then,

$$B = \{(a_1 - 1)^{m_1} \cdot (a_2 - 1)^{m_2} \dots (a_r - 1)^{m_r} \cdot h \mid 0 \leq m_i \leq n_i - 1, h \in H\}$$

is a filtered multiplicative basis of  $KG$  for any field  $K$  of characteristic  $p$ .

If we insist that the elements of  $B$  outside  $\text{rad}(R)$  should be orthogonal idempotents, then we have to require that  $K$  be algebraically closed, as otherwise  $KH$  itself need not have a filtered multiplicative basis  $B$  satisfying

MB3) If  $e, e' \in B \setminus \text{rad}(R)$ ,  $e \neq e'$ , then  $e^2 = e$  and  $e \cdot e' = 0$ .

Observe that, more generally, if  $B_1, B_2$  are filtered multiplicative bases for  $KG_1$  and  $KG_2$ , then  $B_1 \times B_2$  is a filtered multiplicative basis for  $K[G_1 \times G_2]$ .

In the next paragraphs we will examine examples of  $p$ -groups.

If  $G$  is a  $p$ -group, and  $K$  a field of characteristic  $p$ , then  $\text{rad}(KG)$  is the augmentation ideal

$$\text{rad}(KG) = \left\{ \sum_{g \in G} \alpha_g g \mid \sum_{g \in G} \alpha_g = 0 \right\}$$

Note also that in that case, a filtered multiplicative basis  $B$  for  $KG$  necessarily contains 1. Indeed,  $\dim_k \{KG/\text{rad}(KG)\} = 1$ . If  $e \in B$  is the unique element outside  $\text{rad}(KG)$ , then  $e^2 \notin \text{rad}(KG)$  and therefore  $e^2 = e$ . But  $KG$  is local, thus  $e = 1$ . (Alternatively,  $e = 1 + r$  with  $r \in \text{rad}(KG)$  and  $e = e^{p^N} = 1 + r^{p^N} = 1$ .)

Thus for  $p$ -groups, axiom MB3) is automatically satisfied.

§ 2. EXAMPLES WHERE THE GROUND FIELD IS IRRELEVANT

Let  $p$  be a prime number (e.g.  $p=2$ ), and let  $n$  be a natural number,  $n \geq 4$ .

Consider the group  $H_n(p)$  defined by generators and relations:

$$H_n(p) = \langle a, b : a^{p^{n-1}} = 1, b^p = 1, ba = a^{1+p^{n-2}}b \rangle .$$

The group  $H_n(p)$  is a finite  $p$ -group of order  $p^n$ .

PROPOSITION 1. *Let  $K$  be an arbitrary field of characteristic  $p$ . The group algebra  $K[H_n(p)]$  does not possess any filtered multiplicative basis.*

Remark. In contrast, consider the dihedral group

$$D(2^n) = \langle r, s : r^{2^{n-1}} = 1, s^2 = 1, sr = r^{-1}s \rangle .$$

Both  $D(2^n)$  and  $H_n(2)$  are semi-direct products of  $\mathbf{Z}/2^{n-1}\mathbf{Z}$  by  $\mathbf{Z}/2\mathbf{Z}$ . However, a straightforward calculation shows that the set  $B$  consisting of the following elements

$$\begin{aligned} &1, 1 + s, \\ &(r + s)^k, (1 + s)(r + s)^k \quad \text{for } k = 1, \dots, 2^{n-2}, \\ &(r + s)^l(1 + s), (1 + s)(r + s)^l(1 + s) \quad \text{for } l = 1, \dots, 2^{n-2} - 1 \end{aligned}$$

is a filtered multiplicative basis of  $K[D(2^n)]$  for any field  $K$  of characteristic 2.

We proceed to prove Proposition 1. Let  $M = \text{rad } K[H_n(p)]$ . Recall that  $a, b$  are the generators of the defining presentation of  $H_n(p)$ .

LEMMA. *Let  $L_k$  be the set*

$$L_k = \{(1-a)^{k_1}(1-b)^{k_2} \mid 0 \leq k_1 < p^{n-1}, 0 \leq k_2 < p \text{ and } k_1 + k_2 = k\} .$$

Claim: *The classes mod  $M^{k+1}$  of the elements of  $L_k$  form a  $K$ -basis of  $M^k/M^{k+1}$ .*

Proof. We show first, by induction on  $k$ , that the set

$$\{(1-a)^l(1-b)^{k-l} \mid 0 \leq l \leq k\}$$

is a system of  $K$ -generators of  $M^k \text{ mod } M^{k+1}$ .

If  $g, g' \in H_n(p)$ , the identity

$$1 - g \cdot g' = (1-g) + (1-g') - (1-g)(1-g')$$

implies that  $\{(1-a), (1-b)\}$  is a system of  $K$ -generators of  $M \bmod M^2$ .

Suppose by induction that

$$\{(1-a)^l (1-b)^{m-l} \mid 0 \leq l \leq m\}$$

is a  $K$ -generator system of  $M^m \bmod M^{m+1}$ . The set of products  $u_1 \cdot u_2$  with  $u_1 \in M$ ,  $u_2 \in M^m$  generates  $M^{m+1}$  over  $K$ . Thus we have by induction,

$$u_1 = \alpha_1(1-a) + \alpha_2(1-b) \bmod M^2 \quad \text{with} \quad \alpha_1, \alpha_2 \in K,$$

$$u_2 = \sum_{l=0}^m \beta_l (1-a)^l (1-b)^{m-l} \bmod M^{m+1} \quad \text{with} \quad \beta_l \in K.$$

Hence

$$u_1 \cdot u_2 = \sum_{l=0}^m (\alpha_1 \beta_l (1-a)^{l+1} (1-b)^{m-l} + \alpha_2 \beta_l (1-b) (1-a)^l (1-b)^{m-l}) \\ \bmod M^{m+2}.$$

Now,

$$(1-b)(1-a) = 1 - a - b + ba \\ = 1 - a - b + ab - (ab - ba) \\ = (1-a)(1-b) - (ab - ba).$$

But

$$ab - ba \in M^{p^{n-2}} \subset M^3 \quad (\text{recall } n \geq 4),$$

since

$$ab - ba = ab - a^{1+p^{n-2}} b = (1-a)^{p^{n-2}} ab.$$

It follows that

$$(1-b)(1-a) = (1-a)(1-b) \bmod M^3$$

and therefore

$$(1-b)(1-a)^l (1-b)^{m-l} = (1-a)^l (1-b)^{m-l+1} \bmod M^{m+2}.$$

Consequently,

$$u_1 \cdot u_2 = \sum_{l=0}^m (\alpha_1 \beta_l (1-a)^{l+1} (1-b)^{m-l} + \alpha_2 \beta_l (1-a)^l (1-b)^{m-l+1}) \bmod M^{m+2}$$

and the set

$$L = \{(1-a)^{k_1} (1-b)^{k_2} \mid 0 \leq k_1 < p^{n-1}, 0 \leq k_2 < p\}$$

is a system of  $K$ -generators of  $K[H_n(p)]$ .

Since

$$|L| = p^n = |H_n(p)| = \dim_K K[H_n(p)],$$

it follows that  $L$  is a  $K$ -basis of  $K[H_n(p)]$ .

We have just proved that

$$L_k = \{(1-a)^{k_1} (1-b)^{k_2} \mid 0 \leq k_1 < p^{n-1}, 0 \leq k_2 < p, k_1 + k_2 = k\}$$

generates  $M^k \bmod M^{k+1}$ .

We have to prove that  $L_k$  is linearly free over  $K$ . If  $\sum_{t \in L_k} \alpha_t t = 0 \bmod M^{k+1}$  where  $\alpha_t \in K$  then we can write  $\sum_{t \in L_k} \alpha_t t = \sum_{\substack{s \in UL_l \\ l > k}} \beta_s s$  where  $\beta_s \in K$ . Consequently  $\alpha_t = 0$  for all  $t$  in  $L_k$  because  $L$  is a  $K$ -basis of  $K[H_n(p)]$ .

We now come to the proof that  $K[H_n(p)]$  has no filtered multiplicative basis.

We proceed by contradiction. If  $B$  were such a basis, consider

$$\{u, v\} = B \cap \{M \setminus M^2\},$$

the set of elements of  $B$  in  $M$  but outside  $M^2$ .

$\{u, v\}$  is a  $K$ -basis of  $M \bmod M^2$ . Also  $K[H_n(p)] = K[u, v]$ , the algebra generated over  $K$  by  $u$  and  $v$ .

We are going to prove:

*Claim:*  $u \cdot v = v \cdot u$

This implies that  $K[H_n(p)] = K[u, v]$  is commutative: Contradiction.

*Proof of the claim.* By the lemma,

$$u = x_1(1-a) + y_1(1-b) \bmod M^2$$

$$v = x_2(1-a) + y_2(1-b) \bmod M^2,$$

where  $x_1, x_2, y_1, y_2 \in K$  and  $x_1 y_2 - x_2 y_1 \neq 0$ .

Now,

$$u \cdot v = x_1 x_2 (1-a)^2 + y_1 y_2 (1-b)^2 + (x_1 y_2 + x_2 y_1) (1-a) (1-b) \bmod M^3$$

$$v \cdot u = x_1 x_2 (1-a)^2 + y_1 y_2 (1-b)^2 + (x_1 y_2 + x_2 y_1) (1-a) (1-b) \bmod M^3$$

since

$$(1-a)(1-b) = (1-b)(1-a) \pmod{M^3}.$$

We know that  $(1-a)^2, (1-b)^2, (1-a)(1-b)$  forms a  $K$ -basis of  $M^2/M^3$ . Hence  $u \cdot v \neq 0$  and  $v \cdot u \neq 0 \pmod{M^3}$ . Otherwise

$$x_1x_2 = y_1y_2 = x_1y_2 + x_2y_1 = 0$$

and  $x_1y_2 - x_2y_1 = 0$  contrary to the fact that  $\{u, v\}$  gives a basis of  $M/M^2$ .

Thus  $uv, vu \in B$  satisfy  $uv = vu \pmod{M^3}$  and  $uv \neq 0, vu \neq 0 \pmod{M^3}$ .

It follows that  $uv = vu$ . In fact more generally, if  $u_1, u_2 \in B \setminus M^k$  and  $u_1 = u_2 \pmod{M^k}$  then  $u_1 = u_2$ . Proof:  $B \cap M^k$  is a basis of  $M^k$ , thus  $u_1 - u_2 = \sum_{u \in B \cap M^k} \lambda_u u$ . This is possible only if  $u_1 - u_2 = 0$ .

### § 3. THE GROUP OF QUATERNION UNITS

Let  $Q$  be defined by generators and relations:

$$Q = \langle a, b : a^4 = 1, b^2 = a^2, ab = b^3a \rangle.$$

Set  $i = a, j = b, k = ab$  and  $c = a^2$ . Then

$$Q = \{1, c, i, ci, j, cj, k, ck\}.$$

**PROPOSITION 2.** *Let  $K$  be a field of characteristic 2. The group algebra  $K[Q]$  possesses a filtered multiplicative basis if and only if  $K$  contains a primitive cube root of unity.*

*Proof.* If  $K$  contains a primitive cube root of unity, say  $\omega$ ,  
let

$$B = \{1, u, v, uv, vu, u^2, v^2, u^3\},$$

where

$$\begin{aligned} u &= \omega i + \omega^2 j + k \\ v &= \omega^2 i + \omega j + k. \end{aligned}$$

It is easily verified that  $B$  is a filtered multiplicative basis.

Conversely, suppose that  $K[Q]$  possesses a filtered multiplicative basis  $B$ .

Observe that  $\{1+i, 1+j\}$  is a  $K$ -basis of  $M/M^2$ , where again  $M = \text{rad } K[Q]$ . Also  $\{1+c, 1+i+j+k\}$  is a  $K$ -basis of  $M^2/M^3$ . Since  $B \cap (M/M^2) = \{u, v\}$  must be a  $K$ -basis of  $M \text{ mod } M^2$ , we have

$$u = x_1(1+i) + y_1(1+j) \text{ mod } M^2,$$

$$v = x_2(1+i) + y_2(1+j) \text{ mod } M^2,$$

with  $x_1, y_1, x_2, y_2 \in K$  and  $x_1y_2 + x_2y_1 \neq 0$ .

Now

$$\begin{aligned} u \cdot v &= (x_1x_2 + y_1y_2 + x_2y_1)(1+c) \\ &\quad + (x_1y_2 + x_2y_1)(1+i+j+k) \text{ mod } M^3, \end{aligned}$$

$$\begin{aligned} v \cdot u &= (x_1x_2 + y_1y_2 + x_1y_2)(1+c) \\ &\quad + (x_1y_2 + x_2y_1)(1+i+j+k) \text{ mod } M^3. \end{aligned}$$

Therefore,

$$u \cdot v + v \cdot u = (x_1y_2 + x_2y_1)(1+c) \text{ mod } M^3 \neq 0 \text{ mod } M^3,$$

and so  $u \cdot v \neq v \cdot u$ .

We must have  $uv \in B \cap (M^2 \setminus M^3)$  since the  $(1+i+j+k)$ -coordinate of  $u \cdot v$  is non-zero. Similarly  $v \cdot u \in B \cap (M^2 \setminus M^3)$ . But  $\dim(M^2/M^3) = 2$  and so

$$B \cap (M^2 \setminus M^3) = \{uv, vu\}.$$

Consider the element  $u^2 \in M^2$ . Either  $u^2 = uv$  or  $u^2 = vu$  or  $u^2 \in M^3$ . But  $u^2 = (x_1^2 + y_1^2 + x_1y_1)(1+c) \text{ mod } M^3$ .

Since the  $(1+i+j+k)$ -coordinate of  $u^2$  is 0, we have  $u^2 \neq uv$ ,  $u^2 \neq vu$ .

Hence  $u^2 \in M^3$ . This implies  $u^2 = 0$ , and it follows that the quadratic form  $x_1^2 + y_1^2 + x_1y_1$  represents 0 non-trivially in  $K$  and  $w = y_1/x_1$  is a primitive cube root of unity in  $K$ .

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