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Let  $C$  and  $C_1$  be two simple closed curves on  $F$  representing respectively the classes  $\mathbf{C}$  and  $\mathbf{C}_1$ , in such a way that  $C$  and  $C_1$  are in a position of minimum-intersection number.

Consider a neighborhood of the union of  $C$  and  $C_1$  obtained by taking the union of a thin tubular neighborhood of each of these curves, and let  $C_2$  denote the collection of those boundary curves of this neighborhood which are not null-homotopic.

Suppose first of all that  $C_2$  is not empty. Then we have  $s_1(C_2) = C_2$  and  $s_2(C_2) = C_2$ . (To see this, one can represent  $s_1$  (respectively  $s_2$ ) by an isometry of some hyperbolic metric, and then consider the geodesics  $g$  and  $g_1$  in the classes of  $C$  and  $C_1$ . The isometry preserves the geodesics union  $g \cup g_1$  and therefore it preserves an imbedded  $\varepsilon$ -neighborhood of that subset, and the boundary of the neighborhood). In this case,  $s_1$  and  $s_2$  have a common fixed point in **PMF**.

Suppose now that  $C_2$  is empty. We have  $s_1 \circ s_2(\mathbf{C}) = \mathbf{C}$  and  $s_1 \circ s_2(\mathbf{C}_1) = \mathbf{C}_1$ , and  $\mathbf{C}$  and  $\mathbf{C}_1$  have the property that for any element  $\mathbf{F}$  in **MF**, we have either  $i(\mathbf{F}, \mathbf{C}) \neq 0$  or  $i(\mathbf{F}, \mathbf{C}_1) \neq 0$ .

By assumption,  $s_1 \circ s_2$  is reducible. Let  $n$  be an integer s.t. the map  $(s_1 \circ s_2)^n$  preserves each component of the surface  $F$  cut along the reducing curve.

The mapping class  $(s_1 \circ s_2)^n$  cannot have any pseudo-Anosov component, since if it had one, and if  $\mathbf{F}^u$  denotes the class of the unstable foliation of that component, we have either  $i(\mathbf{F}^u, \mathbf{C}) \neq 0$  or  $i(\mathbf{F}^u, \mathbf{C}_1) \neq 0$ . By the dynamics of a pseudo-Anosov (component) map on measured foliations space, the two classes of curves cannot be fixed by  $s_1 \circ s_2$ . Therefore,  $s_1 \circ s_2$  cannot have pseudo-Anosov components.

So  $(s_1 \circ s_2)^n$  has only finite order components.

By the same argument,  $(s_1 \circ s_2)^n$  cannot have a non-trivial Dehn twist along a component of its reducing curve.

Therefore,  $s_1 \circ s_2$  has only periodic components with no non-trivial Dehn twists along the reducing curve, so it is globally periodic, i.e. of finite order, a contradiction.

We conclude that  $s_1 \circ s_2$  is pseudo-Anosov. This proves theorem 2.

## 5. REMARKS AND EXAMPLES

1. We can easily classify now the structure of the group generated by two involutions:

Given the two involutions  $s_1$  and  $s_2$  of  $M(F)$ , the subgroup  $G$  they generate is an order-2 extension of the cyclic subgroup generated by  $s_1 \circ s_2$ . The elements of  $G$  that are not in that subgroup are all conjugate to  $s_1$  or  $s_2$ . If  $s_1$  and  $s_2$  have a common fixed point in  $\mathbf{T}$ , the subgroup that they generate is finite. Otherwise, it is isomorphic to the infinite dihedral group  $Z_2 * Z_2$ .

2. In closing, we wish to point out that all three cases of Theorem 2 do in fact occur in every genus: To see that  $s_1 \circ s_2$  can be of finite order we can take  $s_1$  to be an horizontal rotation as in figure 2 and  $s_2$  to be a vertical rotation as in figure 3. Since these rotations commute,  $s_1 \circ s_2$  is an involution. (This example obviously generalizes to genus greater than two.)

To see that  $s_1 \circ s_2$  can be reducible of infinite order, we can take  $s_1$  to be a vertical rotation as in figure 3 and let  $s_2 = s_1 \circ t_1 \circ t_b^{-1}$ . Now  $s_2$  is an involution by equation (1):

$$(17) \quad (s_2)^2 = s_1 \circ t_1 \circ t_b^{-1} \circ s_1 \circ t_1 \circ t_b^{-1} = t_b \circ t_1^{-1} \circ t_1 \circ t_b^{-1} = 1.$$

Moreover,  $s_1 \circ s_2 = t_1 \circ t_b^{-1} - 1$  which is a reducible map of infinite order. (Again, this example obviously generalizes to higher genera.)

To see that  $s_1 \circ s_2$  can be pseudo-Anosov we can make a similar construction. Let  $s_1$  be an involution. Suppose that  $A$  is a family of disjoint nontrivial simple closed curves. Let  $B = s_1(A)$ . Now suppose that  $A$  and  $B$  fill up  $F$ . Let  $t_A$  be the product of the Dehn twists about the components of  $A$  and  $t_B$  be the corresponding product associated to  $B$ . Let  $s_2 = s_1 \circ t_A \circ t_B^{-1}$ . As in the reducible case just described,  $s_2$  is an involution. Furthermore,  $s_1 \circ s_2 = t_A \circ t_B^{-1}$ , which is a pseudo-Anosov map by an algorithm of Long's [6] generalizing Thurston's algorithm described in [4]. An example of this construction of case (iii) of Theorem 2 is depicted in figure 11, where  $s_1$  is again the vertical rotation. (Again, this example easily generalizes.)

Alternatively, one can give a nonconstructive argument as follows. Let  $s_1$  be a vertical rotation as in figure 3. Since  $s_1(a_1) = b$ , we know that  $s_1$

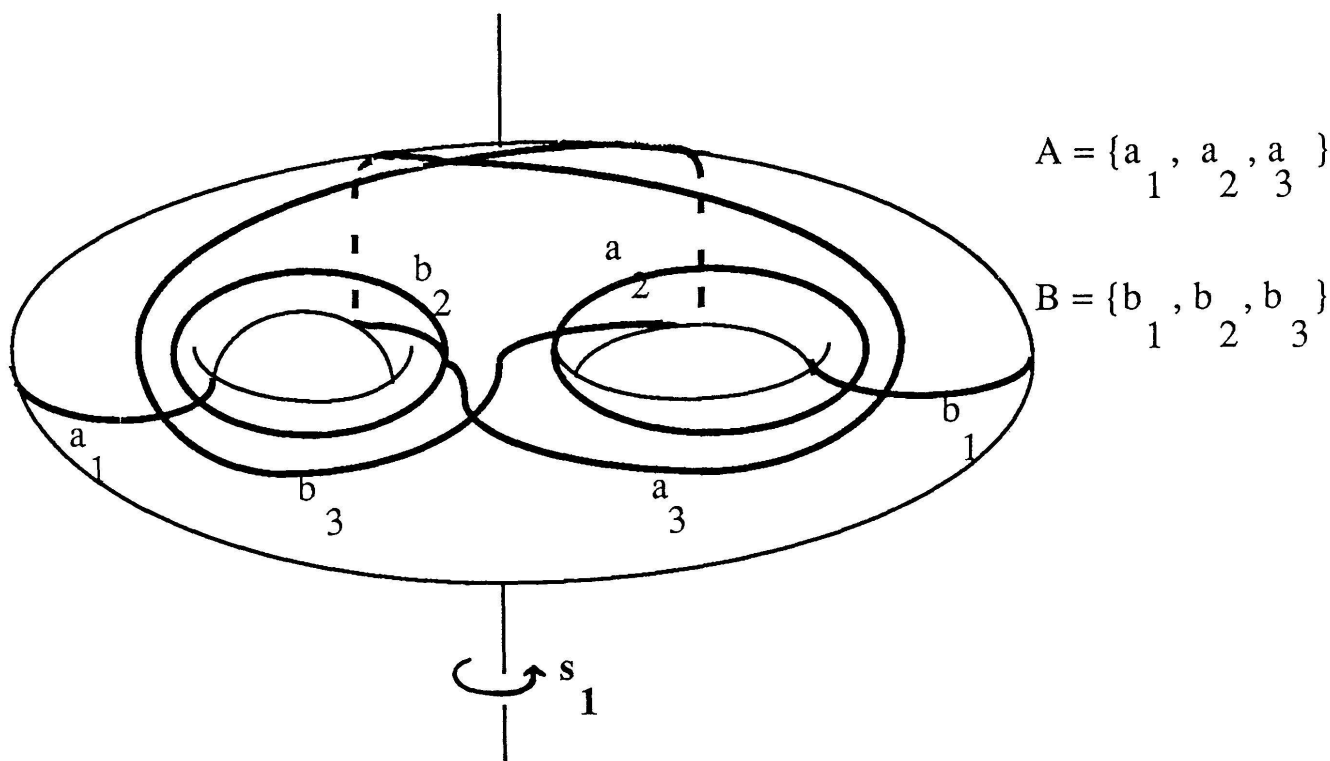


FIGURE 11.

is not in  $\text{Fix}(s_1)$ . On the other hand,  $\text{Fix}(s_1)$  is clearly a closed set. Hence, we may find an open neighborhood of  $a_1$  in  $\mathbf{T} \cup \mathbf{PMF}$ ,  $U$ , such that  $U$  avoids  $\text{Fix}(s_1)$ . Now, we may find a pseudo-Anosov,  $f$ , both of whose fixed points lie in  $U$ . (For example, this can be achieved by conjugating any given pseudo-Anosov by a sufficiently high power of  $t_1$ .) Since  $\text{Fix}(s_1)$  is a compact set which avoids the repelling fixed point of  $f$ , it follows from the well known behavior of pseudo-Anosov maps on  $\mathbf{T} \cup \mathbf{PMF}$  that  $f^n(\text{Fix}(s_1))$  is contained in  $U$  for sufficiently large  $n$ . Choose  $n$  subject to this condition and let  $s_2 = f^n \circ s_1 \circ f^{-n}$ . Finally, since  $\text{Fix}(s_2)$  is equal to  $f^n(\text{Fix}(s_1))$ , it follows from Theorem 2 that  $s_1 \circ s_2$  is pseudo-Anosov.