

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 33 (1987)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** INVOLUTIONS IN SURFACE MAPPING CLASS GROUPS  
**Autor:** McCarthy, John / Papadopoulos, Athanase  
**Kapitel:** 4. Products of involutions  
**DOI:** <https://doi.org/10.5169/seals-87897>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 27.04.2026

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

## 4. PRODUCTS OF INVOLUTIONS

Let  $s_1$  and  $s_2$  be two involutions. We are interested in the type of the element  $s_1 \circ s_2$ . This type will be seen to depend upon the intersection of the two sets  $\text{Fix}(s_1)$  and  $\text{Fix}(s_2)$ , where  $\text{Fix}(s_i)$  denotes the fixed point set of  $s_i$  in the closed ball  $\mathbf{T} \cup \mathbf{PMF}$ .

## THEOREM 2.

(i)  $s_1 \circ s_2$  is of finite order if and only if  $\text{Fix}(s_1)$  and  $\text{Fix}(s_2)$  have a common point in  $\mathbf{T}$ .

(ii) Suppose that  $s_1 \circ s_2$  is not of finite order. If  $\text{Fix}(s_1) \cap \text{Fix}(s_2) \neq \emptyset$ , then  $s_1 \circ s_2$  is reducible.

(iii)  $s_1 \circ s_2$  is pseudo-Anosov if and only if  $\text{Fix}(s_1)$  and  $\text{Fix}(s_2)$  have empty intersection.

*Proof.* (i) If  $s_1$  and  $s_2$  have a common fixed point in  $\mathbf{T}$ , then  $s_1 \circ s_2$  also fixes this point and is therefore of finite order (cf. [4]).

For the converse, suppose that  $s_1 \circ s_2$  is of finite order. Then by ([2], remarque p. 67), there is a point  $m$  in Teichmüller space such that  $m$  is fixed by  $s_1 \circ s_2$ .

The mapping classes  $s_1$  and  $s_2$  being involutions, we have  $s_1(m) = s_2(m)$ .

Now Teichmüller space has a metric, the Teichmüller metric (cf. [1]), for which the mapping class group acts by isometries. By Teichmüller's theorem, any two points in  $\mathbf{T}$  can be joined by a unique geodesic. Each of the mapping classes  $s_1$  and  $s_2$  interchanges the points  $m$  and  $s_1(m)$ . Therefore,  $s_1$  and  $s_2$  fix the point which is at equal distance from  $m$  and  $s_1(m)$ , on the Teichmüller geodesic joining these points.

(ii) Let  $\mathbf{F}$  be a common fixed point of  $s_1$  and  $s_2$  in  $\mathbf{PMF}$ . There exist two positive real numbers  $x_1$  and  $x_2$  such that if  $f$  is an element of  $\mathbf{MF}$  in the class  $\mathbf{F}$ , then  $s_1(f) = x_1 \cdot f$  and  $s_2(f) = x_2 \cdot f$ .

As  $s_1$  and  $s_2$  are of finite order, we have  $x_1$  and  $x_2 = 1$ , so  $s_1 \circ s_2(f) = f$ . By ([2], exposé 9, §.III et IV), either  $s_1 \circ s_2$  is of finite order or it is reducible.

(iii) Suppose that  $\text{Fix}(s_1) \cap \text{Fix}(s_2)$  is empty. By (i),  $s_1 \circ s_2$  is not of finite order. Suppose that it is reducible, and let  $\mathbf{C}$  be the element of  $\mathbf{MF}$  corresponding to the class of the reducing curve. We have  $s_1(\mathbf{C}) = s_2(\mathbf{C})$ . Let  $\mathbf{C}_1$  denote the equivalence class  $s_1(\mathbf{C})$ .

Let  $C$  and  $C_1$  be two simple closed curves on  $F$  representing respectively the classes  $\mathbf{C}$  and  $\mathbf{C}_1$ , in such a way that  $C$  and  $C_1$  are in a position of minimum-intersection number.

Consider a neighborhood of the union of  $C$  and  $C_1$  obtained by taking the union of a thin tubular neighborhood of each of these curves, and let  $C_2$  denote the collection of those boundary curves of this neighborhood which are not null-homotopic.

Suppose first of all that  $C_2$  is not empty. Then we have  $s_1(C_2) = C_2$  and  $s_2(C_2) = C_2$ . (To see this, one can represent  $s_1$  (respectively  $s_2$ ) by an isometry of some hyperbolic metric, and then consider the geodesics  $g$  and  $g_1$  in the classes of  $C$  and  $C_1$ . The isometry preserves the geodesics union  $g \cup g_1$  and therefore it preserves an imbedded  $\varepsilon$ -neighborhood of that subset, and the boundary of the neighborhood). In this case,  $s_1$  and  $s_2$  have a common fixed point in **PMF**.

Suppose now that  $C_2$  is empty. We have  $s_1 \circ s_2(\mathbf{C}) = \mathbf{C}$  and  $s_1 \circ s_2(\mathbf{C}_1) = \mathbf{C}_1$ , and  $\mathbf{C}$  and  $\mathbf{C}_1$  have the property that for any element  $\mathbf{F}$  in **MF**, we have either  $i(\mathbf{F}, \mathbf{C}) \neq 0$  or  $i(\mathbf{F}, \mathbf{C}_1) \neq 0$ .

By assumption,  $s_1 \circ s_2$  is reducible. Let  $n$  be an integer s.t. the map  $(s_1 \circ s_2)^n$  preserves each component of the surface  $F$  cut along the reducing curve.

The mapping class  $(s_1 \circ s_2)^n$  cannot have any pseudo-Anosov component, since if it had one, and if  $\mathbf{F}^u$  denotes the class of the unstable foliation of that component, we have either  $i(\mathbf{F}^u, \mathbf{C}) \neq 0$  or  $i(\mathbf{F}^u, \mathbf{C}_1) \neq 0$ . By the dynamics of a pseudo-Anosov (component) map on measured foliations space, the two classes of curves cannot be fixed by  $s_1 \circ s_2$ . Therefore,  $s_1 \circ s_2$  cannot have pseudo-Anosov components.

So  $(s_1 \circ s_2)^n$  has only finite order components.

By the same argument,  $(s_1 \circ s_2)^n$  cannot have a non-trivial Dehn twist along a component of its reducing curve.

Therefore,  $s_1 \circ s_2$  has only periodic components with no non-trivial Dehn twists along the reducing curve, so it is globally periodic, i.e. of finite order, a contradiction.

We conclude that  $s_1 \circ s_2$  is pseudo-Anosov. This proves theorem 2.

## 5. REMARKS AND EXAMPLES

1. We can easily classify now the structure of the group generated by two involutions: