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## 4. PRODUCTS OF INVOLUTIONS

Let  $s_1$  and  $s_2$  be two involutions. We are interested in the type of the element  $s_1 \circ s_2$ . This type will be seen to depend upon the intersection of the two sets  $\text{Fix}(s_1)$  and  $\text{Fix}(s_2)$ , where  $\text{Fix}(s_i)$  denotes the fixed point set of  $s_i$  in the closed ball  $\mathbf{T} \cup \mathbf{PMF}$ .

## THEOREM 2.

- (i)  $s_1 \circ s_2$  is of finite order if and only if  $\text{Fix}(s_1)$  and  $\text{Fix}(s_2)$  have a common point in  $\mathbf{T}$ .
- (ii) Suppose that  $s_1 \circ s_2$  is not of finite order. If  $\text{Fix}(s_1) \cap \text{Fix}(s_2) \neq \emptyset$ , then  $s_1 \circ s_2$  is reducible.
- (iii)  $s_1 \circ s_2$  is pseudo-Anosov if and only if  $\text{Fix}(s_1)$  and  $\text{Fix}(s_2)$  have empty intersection.

*Proof.* (i) If  $s_1$  and  $s_2$  have a common fixed point in  $\mathbf{T}$ , then  $s_1 \circ s_2$  also fixes this point and is therefore of finite order (cf. [4]).

For the converse, suppose that  $s_1 \circ s_2$  is of finite order. Then by ([2], remarque p. 67), there is a point  $m$  in Teichmüller space such that  $m$  is fixed by  $s_1 \circ s_2$ .

The mapping classes  $s_1$  and  $s_2$  being involutions, we have  $s_1(m) = s_2(m)$ .

Now Teichmüller space has a metric, the Teichmüller metric (cf. [1]), for which the mapping class group acts by isometries. By Teichmüller's theorem, any two points in  $\mathbf{T}$  can be joined by a unique geodesic. Each of the mapping classes  $s_1$  and  $s_2$  interchanges the points  $m$  and  $s_1(m)$ . Therefore,  $s_1$  and  $s_2$  fix the point which is at equal distance from  $m$  and  $s_1(m)$ , on the Teichmüller geodesic joining these points.

(ii) Let  $\mathbf{F}$  be a common fixed point of  $s_1$  and  $s_2$  in  $\mathbf{PMF}$ . There exist two positive real numbers  $x_1$  and  $x_2$  such that if  $f$  is an element of  $\mathbf{MF}$  in the class  $\mathbf{F}$ , then  $s_1(f) = x_1 \cdot f$  and  $s_2(f) = x_2 \cdot f$ .

As  $s_1$  and  $s_2$  are of finite order, we have  $x_1$  and  $x_2 = 1$ , so  $s_1 \circ s_2(f) = f$ . By ([2], exposé 9, §.III et IV), either  $s_1 \circ s_2$  is of finite order or it is reducible.

(iii) Suppose that  $\text{Fix}(s_1) \cap \text{Fix}(s_2)$  is empty. By (i),  $s_1 \circ s_2$  is not of finite order. Suppose that it is reducible, and let  $\mathbf{C}$  be the element of  $\mathbf{MF}$  corresponding to the class of the reducing curve. We have  $s_1(\mathbf{C}) = s_2(\mathbf{C})$ . Let  $\mathbf{C}_1$  denote the equivalence class  $s_1(\mathbf{C})$ .

Let  $C$  and  $C_1$  be two simple closed curves on  $F$  representing respectively the classes  $\mathbf{C}$  and  $\mathbf{C}_1$ , in such a way that  $C$  and  $C_1$  are in a position of minimum-intersection number.

Consider a neighborhood of the union of  $C$  and  $C_1$  obtained by taking the union of a thin tubular neighborhood of each of these curves, and let  $C_2$  denote the collection of those boundary curves of this neighborhood which are not null-homotopic.

Suppose first of all that  $C_2$  is not empty. Then we have  $s_1(C_2) = C_2$  and  $s_2(C_2) = C_2$ . (To see this, one can represent  $s_1$  (respectively  $s_2$ ) by an isometry of some hyperbolic metric, and then consider the geodesics  $g$  and  $g_1$  in the classes of  $C$  and  $C_1$ . The isometry preserves the geodesics union  $g \cup g_1$  and therefore it preserves an imbedded  $\varepsilon$ -neighborhood of that subset, and the boundary of the neighborhood). In this case,  $s_1$  and  $s_2$  have a common fixed point in  $\mathbf{PMF}$ .

Suppose now that  $C_2$  is empty. We have  $s_1 \circ s_2(\mathbf{C}) = \mathbf{C}$  and  $s_1 \circ s_2(\mathbf{C}_1) = \mathbf{C}_1$ , and  $\mathbf{C}$  and  $\mathbf{C}_1$  have the property that for any element  $\mathbf{F}$  in  $\mathbf{MF}$ , we have either  $i(\mathbf{F}, \mathbf{C}) \neq 0$  or  $i(\mathbf{F}, \mathbf{C}_1) \neq 0$ .

By assumption,  $s_1 \circ s_2$  is reducible. Let  $n$  be an integer s.t. the map  $(s_1 \circ s_2)^n$  preserves each component of the surface  $F$  cut along the reducing curve.

The mapping class  $(s_1 \circ s_2)^n$  cannot have any pseudo-Anosov component, since if it had one, and if  $\mathbf{F}^u$  denotes the class of the unstable foliation of that component, we have either  $i(\mathbf{F}^u, \mathbf{C}) \neq 0$  or  $i(\mathbf{F}^u, \mathbf{C}_1) \neq 0$ . By the dynamics of a pseudo-Anosov (component) map on measured foliations space, the two classes of curves cannot be fixed by  $s_1 \circ s_2$ . Therefore,  $s_1 \circ s_2$  cannot have pseudo-Anosov components.

So  $(s_1 \circ s_2)^n$  has only finite order components.

By the same argument,  $(s_1 \circ s_2)^n$  cannot have a non-trivial Dehn twist along a component of its reducing curve.

Therefore,  $s_1 \circ s_2$  has only periodic components with no non-trivial Dehn twists along the reducing curve, so it is globally periodic, i.e. of finite order, a contradiction.

We conclude that  $s_1 \circ s_2$  is pseudo-Anosov. This proves theorem 2.

## 5. REMARKS AND EXAMPLES

1. We can easily classify now the structure of the group generated by two involutions: