

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 33 (1987)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** INVOLUTIONS IN SURFACE MAPPING CLASS GROUPS  
**Autor:** McCarthy, John / Papadopoulos, Athanase  
**Kapitel:** 3. Thurston's classification of mapping classes  
**DOI:** <https://doi.org/10.5169/seals-87897>

#### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

#### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

#### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 12.01.2026

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

By the same reasoning, we deduce that:

$$(15) \quad t_b t_1^{-1} \in \Sigma,$$

$$(16) \quad t_b t_2^{-1} \in \Sigma.$$

Hence, Humphries' generators are all conjugate modulo  $\Sigma$ . This implies that  $M(F_g)/\Sigma$  is cyclic.

If the genus is greater than 2, equation (3) implies that  $\Sigma = M(F_g)$ . Since  $\Sigma$  is contained in  $I(F_g)$ , we see that  $\Sigma = I(F_g) = M(F_g)$ . Hence, theorem 1 is true for genus greater than two.

Now suppose that the genus is two. By equation (12), we conclude that  $i$  belongs to  $\Sigma$ . By lemma 2, it follows that  $\Sigma = I(F_2)$ . On the other hand, by equation (12), we conclude that

$$M(F_2)/\Sigma = \mathbb{Z}_5.$$

This establishes theorem 1 for genus two.

This completes the proof of theorem 1.

### 3. THURSTON'S CLASSIFICATION OF MAPPING CLASSES

The Teichmüller space of  $F$ , denoted by  $\mathbf{T}$ , is the space of hyperbolic metrics on  $F$  up to isometry. It has a natural topology and is homeomorphic to an open ball of dimension  $6g-6+2b$ , where  $g$  is the genus of  $F$  and  $b$  the number of its boundary components.

Thurston's boundary of  $\mathbf{T}$  is the space of projective classes of measured foliations on  $F$ .

A measured foliation is a foliation with isolated singularities of a special type ( $p$ -prong singularities, where  $p$  is any integer  $> 2$ , see figure 10), with a measure on transverse segments which is a Lebesgue-measure, and which is invariant by isotopy of the segment keeping each point on the same leaf.

There's an equivalence relation between measured foliations, generated by isotopy and the operation of collapsing a leaf connecting two singular points.  $\mathbf{MF}$  denotes the space of equivalence classes.

There's a natural action on  $\mathbf{MF}$  by the positive reals;  $\mathbf{PMF}$  is the quotient projective space.  $\mathbf{PMF}$  is homeomorphic to a sphere of dimension  $6g-7+2b$  which constitutes, by Thurston's work, a natural boundary for Teichmüller space.  $M(F)$ , The mapping class group of  $F$ , acts continuously

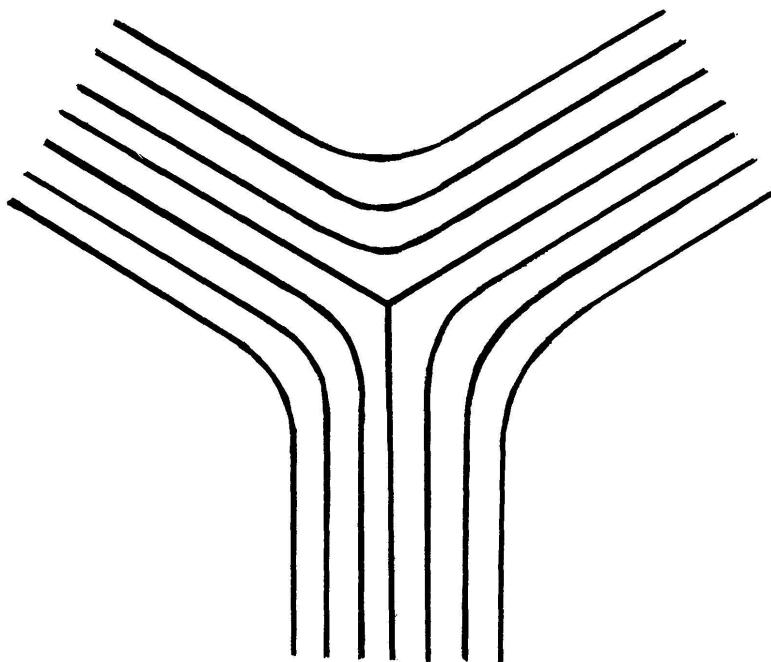


FIGURE 10.

on the closed ball  $T \cup \text{PMF}$ , and Thurston's classification of the elements of  $M(F)$  can be formulated in terms of this action.

If an element of  $M(F)$  has a fixed point in  $T$ , then it is of finite order, i.e. there is an integer  $n$  such that the  $n$ -th iterate of that element is the class of the identity. In fact, there is a representative of this element which is globally periodic of order  $n$ , and which is an isometry of the hyperbolic metric corresponding to that fixed point in  $T$ .

If an element of  $M(F)$  does not have a fixed point in  $T$ , then by the Brouwer fixed-point theorem it has a fixed point in  $\mathbf{PMF}$ .

There are two cases: either this point is the equivalence class of a foliation which has no closed cycles of leaves, and then this element is of *pseudo-Anosov* type, and can be represented by a homeomorphism of the surface which preserves a pair of measured foliations, acting as an expansion with respect to the transverse measure of one of them, and a contraction with respect to the other, or the fixed point in **PMF** is the class of a foliation which has a cycle of leaves; in this case the map is said to be *reducible*. There's an isotopy class of a (nonnecessarily connected) simple closed curve on  $M$  which is preserved by this mapping class, and the mapping class naturally splits into *components*.

We refer to [1] or [4] for the details of this classification.