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PROPOSITION 5.4. *Suppose* e_1, e_2 *and* e_3 *are orthonormal imaginary Cayley numbers with* e_3 *orthogonal to* $e_1 e_2$. *Then there exists a unique automorphism of* Ca *sending* $i = (i, 0) \mapsto e_1$, $j = (j, 0) \mapsto e_2$ *and* $\varepsilon = (0, 1) \mapsto e_3$.

This follows from three applications of Proposition 5.1.

From Proposition 5.4, one concludes that the group of all automorphisms of the Cayley numbers (a Lie group known as G_2) is 14-dimensional.

6. THE HOPF FIBRATION $S^7 \hookrightarrow S^{15} \to S^8$

Choose orthonormal coordinates in R^{16} and identify it with Cayley 2-space Ca^2. In Ca^2 consider subsets of the form

$$L_m = \{(u, mu) : u \in Ca\} \quad \text{for each} \quad m \in Ca\,,$$
$$L_\infty = \{(0, v) : v \in Ca\}\,.$$

They are 8-dimensional real linear subspaces of R^{16}, but *not* Cayley subspaces of Ca^2 because they are not closed under Cayley multiplication. This is the effect of the nonassociativity of the Cayley numbers. Nevertheless, we call L_m and L_∞ *Cayley lines* for simplicity.

We need to check that these Cayley lines fill out Ca^2, with any two meeting only at the origin. Given $(u, v) \in Ca^2$, if $u = 0$ then this point is on the Cayley line L_∞. If $u \neq 0$, let $m = v u^{-1}$. Then $m u = (v u^{-1}) u = v$ by Fact 3 of the preceding section. Hence the point (u, v) lies on the Cayley line L_m. Thus the Cayley lines fill out Ca^2.

Clearly L_∞ meets each other Cayley line only at the origin. And if the point (u, v), with $u \neq 0$, lies on the Cayley lines L_m and L_n, then $v = m u = n u$. Hence $m = n$. Thus any two Cayley lines meet only at the origin.

The unit 7-spheres on these Cayley lines then define for us the *Hopf fibration* $S^7 \hookrightarrow S^{15} \to S^8$. Note that the base space is clearly homeomorphic to an 8-sphere, since there is one Cayley line for each Cayley number m, and one for the number ∞.

In a similar fashion, if we start with any k-dimensional normed division algebra K, we obtain a Hopf fibration

$$S^{k-1} \hookrightarrow S^{2k-1} \to S^k\,.$$

Note by Hurwitz's theorem that K is isomorphic to R, C, H or Ca, so there are really no new cases.

PROPOSITION 6.1. *The Hopf 7-spheres on S^{15} are parallel to one another.*

We must show that the 8-planes

$$P = L_v = \{(u, vu)\} \quad \text{and} \quad Q = L_w = \{(u, wu)\}$$

intersect S^{15} in parallel great 7-spheres.

Let the vectors $e_i, i = 1, ..., 8$ form an orthonormal basis for Ca. Then the vectors $(e_i, v\, e_i)$, $i = 1, ..., 8$ form an orthogonal basis for P, with each vector having length $(1+|v|^2)^{1/2}$. This is an immediate consequence of Fact 1 from the preceding section.

Likewise, the vectors $(e_j, w\, e_j)$, $j = 1, ..., 8$ form an orthogonal basis for Q, with each vector having length $(1+|w|^2)^{1/2}$.

With respect to these bases, the matrix $A = (a_{ij})$ of orthogonal projection of P to Q is given by

$$a_{ij} = <e_i, e_j> + <v\, e_i, w\, e_j> ,$$

or

$$A = I + B .$$

We want to show that A is conformal, i.e., that

$$A\, A^t = I + B + B^t + B\, B^t = \lambda I .$$

First note that

$$\begin{aligned}(B+B^t)_{ij} &= <ve_i, we_j> + <ve_j, we_i> \\ &= <(v+w)e_i, (v+w)e_j> - <ve_i, ve_j> - <we_i, we_j> \\ &= (|v+w|^2 - |v|^2 - |w|^2)\, <e_i, e_j> \\ &= 2<v, w>\, \delta_{ij} ,\end{aligned}$$

by repeated application of Fact 1 of the preceding section. Thus $B + B^t$ is a multiple of the identity.

Next note that

$$\begin{aligned}(B\, B^t)_{ij} &= \Sigma_r <ve_i, we_r> <ve_j, we_r> \\ &= <ve_i, ve_j>\, |\, w\,|^2 = |\, v\,|^2\, |\, w\,|^2\, \delta_{ij} ,\end{aligned}$$

since $we_r, r = 1, ..., 8$ is an orthogonal basis for Ca with each vector of length $|\, w\,|$. Thus $B\, B^t$ is also a multiple of the identity.

It follows that A is conformal, and hence that the 8-planes $P = L_v$ and $Q = L_w$ intersect S^{15} in parallel great 7-spheres. By continuity, the same is true if one of these planes is L_∞. Thus the Hopf 7-spheres on S^{15} are parallel to one another, as claimed. QED

The Riemannian metric on the base space S^8 which makes the Hopf projection $S^{15} \to S^8$ into a Riemannian submersion is that of a round 8-sphere of radius 1/2, which one sees directly just as in the previous cases.

7. Symmetries of the Hopf fibration $H: S^7 \hookrightarrow S^{15} \to S^8$

Proposition 7.1. *The group G of all symmetries of the Hopf fibration $H: S^7 \hookrightarrow S^{15} \to S^8$ is isomorphic to $Spin(9)$, the simply connected double cover of $SO(9)$.*

The action is as follows:

1) *There is a $g \in G$ inducing any preassigned orientation preserving isometry of the round base S^8, but no orientation reversing ones.*
2) *Given such a g, there is exactly one other symmetry,*

$$-g = \text{antipodal map} \circ g\,,$$

which induces the same action on S^8.

It is likely that Élie Cartan was aware of this result, since in [Ca 2, esp. pp. 424 and 466] he identified $Spin(9)$ as the group of isometries fixing a point in the Cayley projective plane CaP^2. It is not hard to see that this is the same as the group of symmetries of our Hopf fibration. The symmetry groups of the other Hopf fibrations can likewise be identified with the groups of isometries fixing a point in complex and quaternionic projective spaces, also known to Cartan.

We give the proof of Proposition 7.1 in a series of lemmas.

Lemma 7.2. *The only symmetries which take each fibre to itself are the identity and the antipodal map.*

Suppose $B: R^{16} \to R^{16}$ is such a symmetry. Since B maps

$$L_0 = \{(u, 0)\}\,, \; L_\infty = \{(0, v)\} \quad \text{and} \quad L_1 = \{(u, u)\}$$

into themselves, we must have

$$B(u, v) = (A(u), A(v))$$

for some $A \in O(8)$. Since B maps $L_m = \{(u, mu)\}$ into itself, we get