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Kapitel: §11. PROOF OF THE THEOREMS OF L. KAUFFMAN AND K. MURASUGI

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§ 11. Proof of the theorems of L. Kauffman and K. Murasugi

Let Γ be an unoriented link projection in S^2. We shall always suppose that the image is connected, to avoid unnecessary complications. Observe that all projections of an unsplittable link have this property.

We consider the chessboard associated to Γ. To the shaded regions we associate a graph $\Sigma \subset S^2$ in the following way: In each shaded region we select a point which will be a vertex of Σ. If two shaded regions meet at a double point of Γ, we draw an edge joining the two vertices through the double point. (If the two regions are not distinct, we will get a loop.)

We proceed in the same way with the unshaded (lightened) regions, to obtain another graph $\Lambda \subset S^2$.

Notice that, if c is the number of double points of Γ and if R is the number of regions determined by Γ, one has $R = c + 2$. This is an immediate consequence of Euler formula and the fact that the image of Γ is a quadrivalent graph.

Now, let L be an unoriented link diagram and write Γ for the underlying link projection.

Let S be a state of L. We shall associate to S a subgraph Σ_S of Σ and a subgraph Λ_S of Λ in the following way:

(i) Σ_S contains all the vertices of Σ.

(ii) Λ_S contains all the vertices of Λ.

(iii) At each double point of Γ, one edge of Λ and one edge of Σ cross each other. We keep the edge which joins the two regions which are connected by the choice (marker) of S at the crossing point and we discard the other edge.

Lemma 11.1. *Σ_S is a deformation retract of $S^2 - \Lambda_S$ and Λ_S is a deformation retract of $S^2 - \Sigma_S$. In other words, Σ_S and Λ_S are duals in S^2 in the sense of J. H. C. Whitehead.*

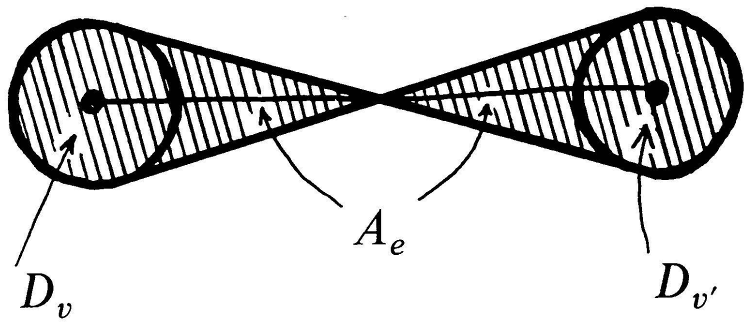
Let Γ_S be the configuration of disjoint simple closed curves in S^2 obtained by cutting and glueing Γ at each crossing point according to the indication given by S. By definition, $|S|$ is the number of connected components of Γ_S.

Lemma 11.2. *Γ_S is the boundary of a regular neighborhood of Σ_S in S^2.*

As Σ_S and Λ_S are Whitehead duals, we can replace Σ_S by Λ_S if we wish.

Proof of lemmas 11.1 and 11.2. Let us observe that we can recapture from Σ the union of the shaded regions in the chessboard by the following procedure:

1) Choose a small disc D_v around each vertex v of Σ.
2) For each edge e in Σ, choose a double apex A_e like in the picture:



The union $\bigcup_v D_v \cup \bigcup_e A_e$ is equal, up to an homeomorphism of S^2, to the union of the shaded regions of the chessboard. Its boundary (frontier) is the link projection Γ.

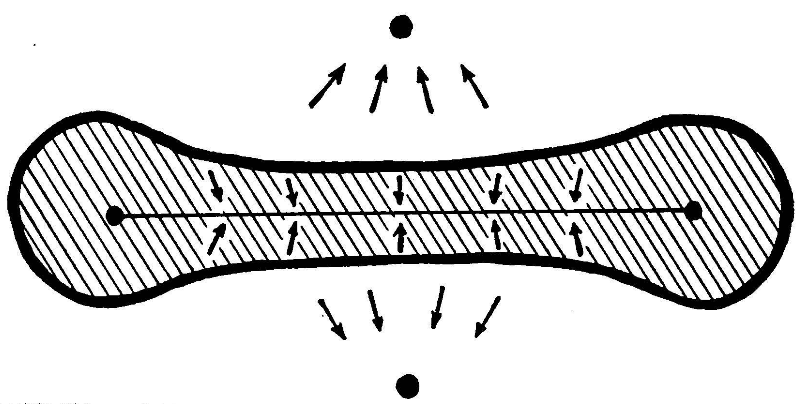
Of course, we could have replaced everywhere in the construction "shaded" by "lightened".

Now, let S be a state for L. Let P be a double point of Γ. The cutting and glueing operation associated to S at P will remove the double point P.

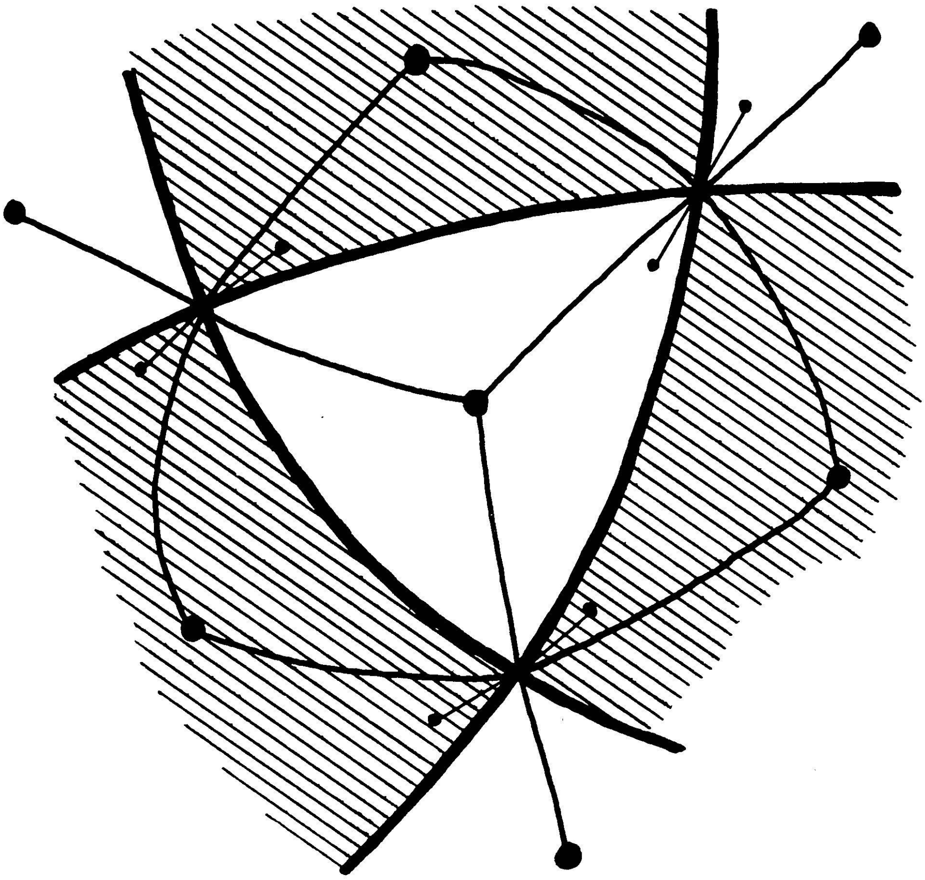
Near P, Γ_S will be the boundary of the shaded surface newly obtained. (And also the boundary of the lightened surface newly obtained.) Suppose, for instance, that the state S chooses at P the marker corresponding to the shaded regions. Then, it is easy to see that, locally around P, the new shaded surface deformation retracts to the edge of Σ_S going through P.

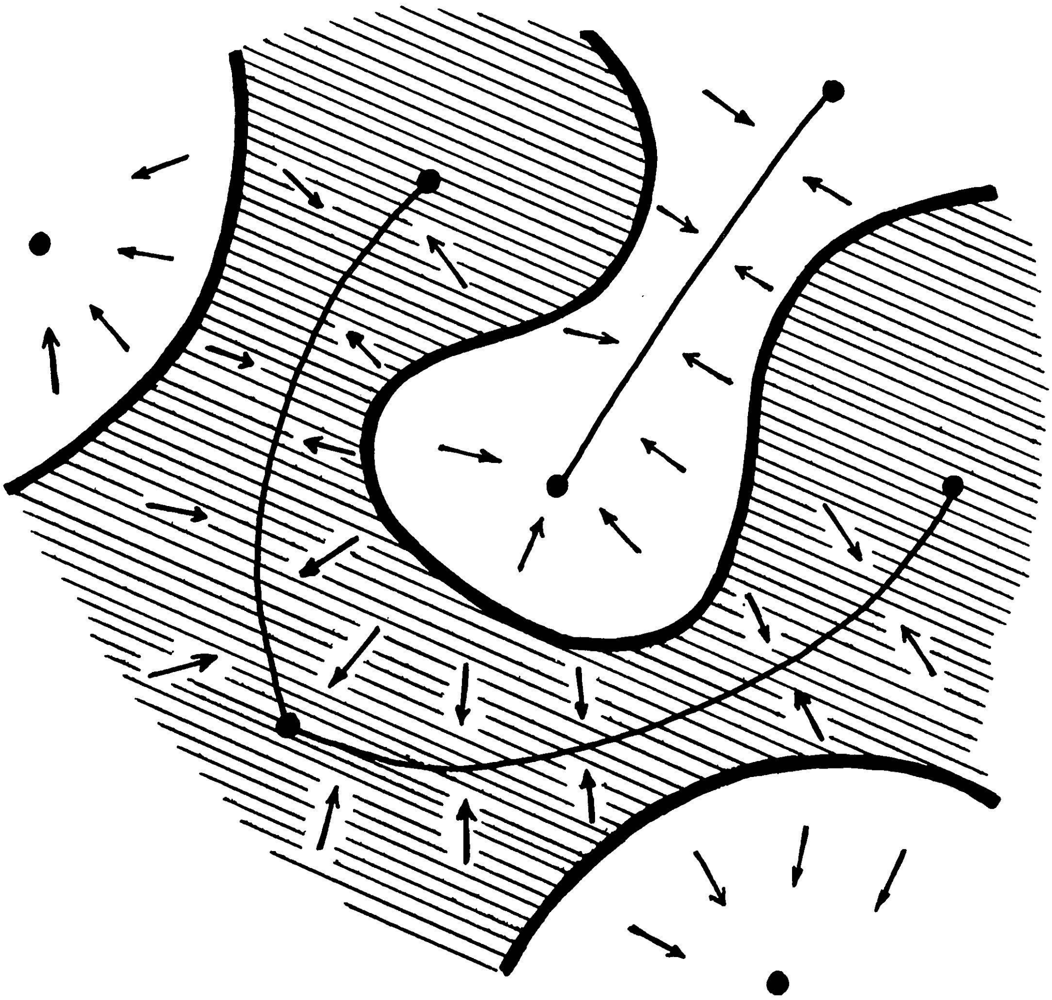
It is also easy to see that, locally around P, the new lightened region deformation retracts on the two vertices of the edge of Λ which has been deleted to obtain Λ_S.

Picture:



The following pictures should help to see what happens locally:

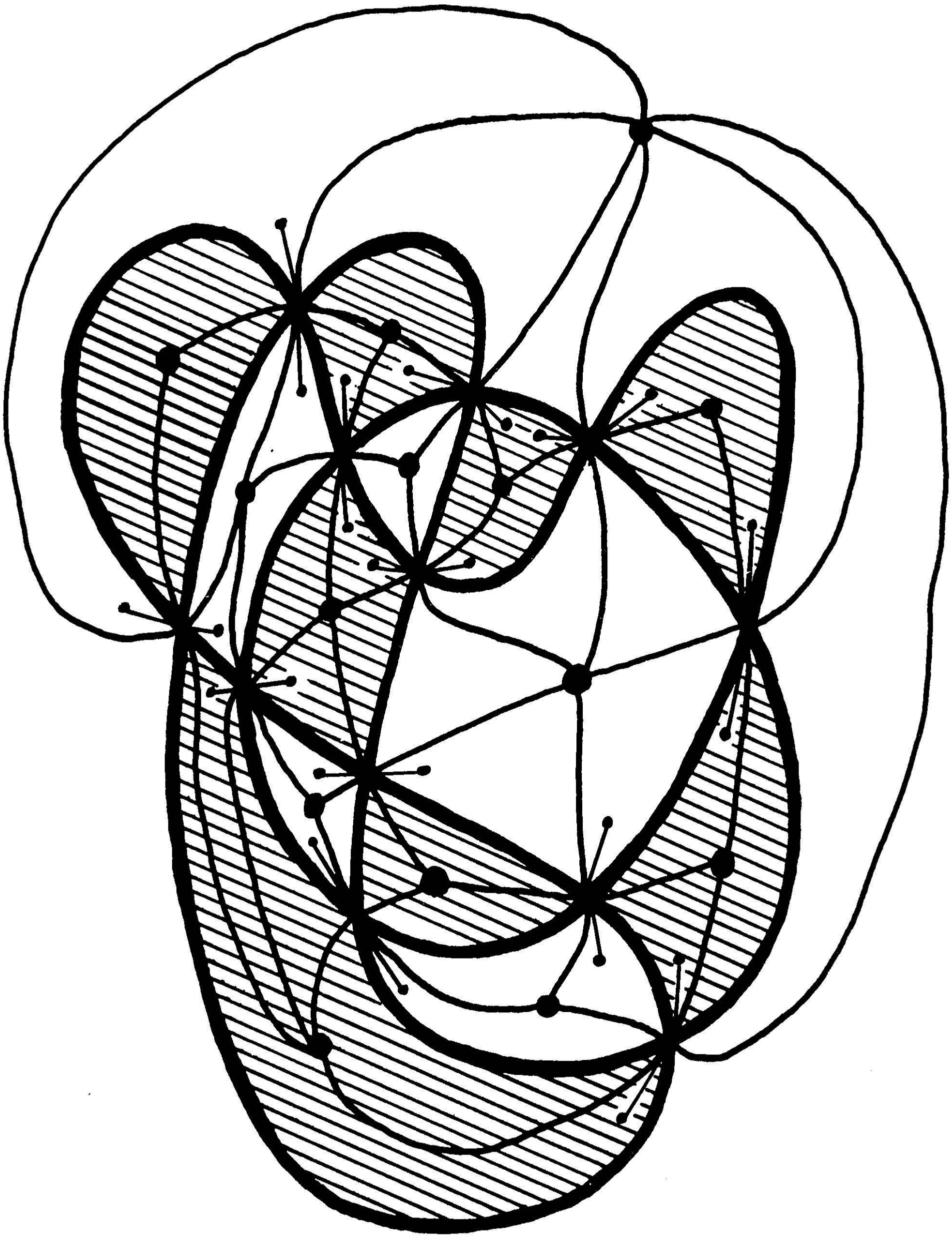


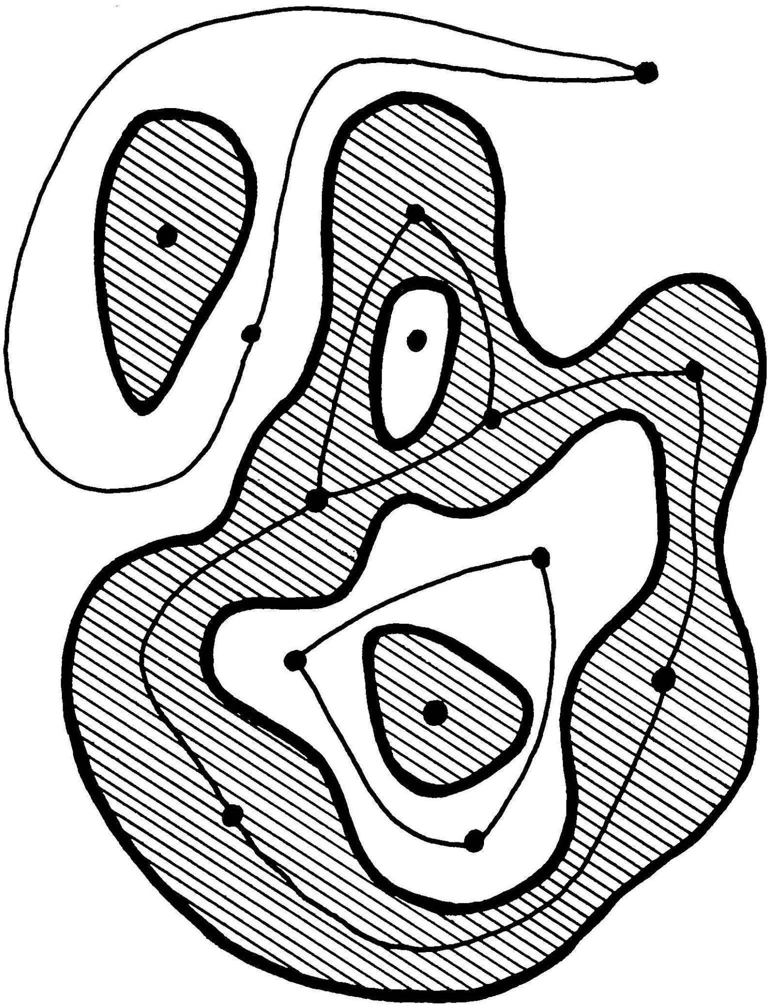


These small deformation retractions can be pieced together in order that globally the newly shaded surface is a regular neighborhood $N(\Sigma_S)$ of Σ_S. In the same way, the newly lightened surface is a regular neighborhood $N(\Lambda_S)$ of Λ_S. The common boundary of $N(\Sigma_S)$ and $N(\Lambda_S)$ is Γ_S.

These constructions are illustrated in the next two pictures. In the first one, a knot projection is shown, with its chessboard, its graphs Σ and Λ. A state S is indicated. The second picture shows Γ_S, Σ_S, Λ_S.

This ends the proofs of lemmas 11.1 and 11.2.





LEMMA 11.3. *Let G be a graph in S^2 and let N be a regular neighborhood of G. Then the number of connected components of ∂N is equal to $b_0(G) + b_1(G)$.*

Notation. $b_i(G)$ denotes the i-th Betti number.

Proof of Lemma 11.3. By Alexander duality:

$$b_0(\partial N) = b_0(N) + b_0(S^2 - N) - 1$$

and

$$b_1(N) = b_0(S^2 - N) - 1 .$$

As N deformation retracts onto G, the result follows.

Recall that the number $|S|$ of connected components of Γ_S is an important ingredient in Kauffman's polynomial.

PROPOSITION 11.4. $|S| = b_1(\Sigma_S) + b_1(\Lambda_S) + 1$.

Note. This proposition is the generalization to any state S of lemma 2 of K. Murasugi's paper [Mu_2].

Proof of proposition 11.4. We know that $|S| = b_0(\Gamma_S)$. Now $\Gamma_S = \partial N(\Sigma_S)$. So, if we apply lemma 11.3 to $G = \Sigma_S$, we get

$$b_0(\Gamma_S) = b_0(\Sigma_S) + b_1(\Sigma_S)\,.$$

As Σ_S and Λ_S are S-duals, Alexander duality implies that

$$b_0(\Sigma_S) = b_1(\Lambda_S) + 1\,.$$

We substitute and the proof is finished.

LEMMA 11.5. *Let G be a connected graph. Let G_1 and G_2 be two subgraphs of G such that* (1) $G = G_1 \cup G_2$. *Let $G_0 = G_1 \cap G_2$ and suppose that* (2) G_0 *contains no edge. Then*

$$b_1(G_1) + b_1(G_2) \leqslant b_1(G)\,.$$

Suppose moreover that (3) G_1 *and* G_2 *have no isolated vertices. Then, one has $b_1(G_1) + b_1(G_2) = b_1(G)$ if and only if each vertex of G_0 is a cut vertex (for the partition associated to G_1 and G_2).*

Consequence: Suppose that G_1 and G_2 have no isolated vertices and that G has no cut vertex at all. Then, if $b_1(G_1) + b_1(G_2) = b_1(G)$ one has that G_1 or G_2 is empty (and $G_2 = G$ or $G_1 = G$).

Before proving lemma 11.5, we make some comments on the notion of cut vertex.

Let v be a vertex of a graph H. Let E_v be the set of edges of H which have v in their boundary. Suppose given a partition of E_v into two non empty classes E_1 and E_2. Then the chopping of H at v is constructed in the following way:

Replace v by two vertices v_1 and v_2 and declare that the edges in E_i will have v_i in their boundary instead of v ($i = 1, 2$).

Definition. v is a *cut vertex* for the partition $E_1 \amalg E_2$ if the chopping of H we just described produces a graph with one more connected component. v is a cut vertex if there exists a partition such that... etc., etc.

Proof of lemma 11.5. The inequality is an immediate consequence of Mayer-Vietoris, using that $b_1(G_0) = 0$.

Now observe that conditions (1) and (2) amount to say that G_1 and G_2 produce a (global) partition of the edges of G in two classes.

Suppose that moreover condition (3) is also satisfied. Let v be a vertex of G_0. Then G_1 and G_2 induce a partition of the set E in two non-empty classes. Hence, the chopping of G at v is well defined.

Write $\hat{G}$ for the graph obtained by chopping G at all the vertices of G_0. Remark that G_1 and G_2 naturally embed in $\hat{G}$. Their union is $\hat{G}$ and their intersection is empty. So

$$b_1(G_1) + b_1(G_2) = b_1(\hat{G}).$$

Now, let $\pi : \hat{G} \to G$ be the natural projection which identifies the pairs of vertices created by the chopping. Remark that identifying two vertices has homologically the same effect as adding a new edge between the two vertices. This replaces π by an inclusion. If we write the end of the homology exact sequence of this inclusion, we see immediately that π induces a monomorphism

$$H_1(\hat{G}) \hookrightarrow H_1(G).$$

The same exact sequence shows that the monomorphism is an isomorphism if and only if each vertex of G_0 is a cut vertex for the partition induced by G_1 and G_2.

End of proof of lemma 11.5.

Notation. Let σ_S be the subgraph of Σ_S obtained by removing the isolated vertices of Σ_S. Let λ_S be the subgraph of Λ_S obtained in the same way.

Of course $b_1(\Sigma_S) = b_1(\sigma_S)$ and $b_1(\Lambda_S) = b_1(\lambda_S)$. So, proposition 11.4 gives $|S| = b_1(\sigma_S) + b_1(\lambda_S) + 1$.

Definition. If S is a state, L. Kauffman calls $\check{S}$ the dual state of S if, at every double point of Γ, the choice opposite to S is made.

It is obvious from the definitions that:

(1) $\sigma_S \cup \sigma_{\check{S}} =$

(2) $\sigma_S \cap \sigma_{\check{S}}$ contains no edge.

(3) σ_S and $\sigma_{\check{S}}$ have no isolated vertices.

The same holds for λ_S and $\lambda_{\check{S}}$ in Λ.

LEMMA 11.6. *$b_1(\Sigma) + 1 = l =$ number of lightened region of the chessboard. $b_1 |\Lambda| + 1 = s =$ number of shaded region in the chessboard.*

Proof. Obvious.

PROPOSITION 11.7. $|S| + |\check{S}| \leqslant l+s = R = c+2$.

Comment. This inequality is the "dual state lemma" of L. Kauffman.

Proof of proposition 11.7.

$$\begin{aligned} |S| + |\check{S}| &\leqslant b_1(\sigma_S) + b_1(\lambda_S) + 1 + b_1(\sigma_{\check{S}}) + b_1(\lambda_{\check{S}}) + 1 \\ &\leqslant b_1(\Sigma) + b_1(\Lambda) + 2 = l+s. \end{aligned} \qquad \text{Q.E.D.}$$

Recall that L is an unoriented link diagram and that Γ is the underlying link projection. Write A for the state defined by choosing "A" at every double point of L. Write B for the state defined by choosing "B" everywhere. Of course, A and B are dual states.

Notation. If S is a state of L, write $\varphi_S(A)$ for the contribution of the state S to the polynomial $<L>$. $\varphi_S(A)$ is an element of $\mathbf{Z}[A^{\pm 1}]$.

Write D_S for the maximal degree of the monomials in $\varphi_S(A)$ and write d_S for the minimal degree.

LEMMA 11.8. *For any state S one has:*

$$D_S \leqslant D_{\mathrm{A}} \quad \text{and} \quad d_{\mathrm{B}} \leqslant d_S .$$

Proof of lemma 11.8. We prove $D_S \leqslant D_{\mathrm{A}}$, the proof of $d_{\mathrm{B}} \leqslant d_S$ being analogous. Write $b = b(S)$ for the number of times "B" has been chosen in the state S. There is a sequence of states:

$\mathrm{A} = S_0, S_1, \ldots, S_b = S$ where S_i differs from S_{i-1} in one double point of L where the "A" has been replaced by a "B".

CLAIM: $D_{S_i} \leqslant D_{S_{i-1}}$.

Obviously the claim implies that $D_S \leqslant D_{\mathrm{A}}$. Come back to the definition of $<L>$. The contribution of S_i is

$$A^{a(S_i)} B^{b(S_i)} d^{|S|-1},$$

where $B = A^{-1}$ and $d = -(A^2+A^{-2})$. The degree of $A^{a(S_i)} B^{b(S_i)}$ is then

$$a(S_i) - b(S_i).$$

So (*) $a(S_i) - b(S_i) = a(S_{i-1}) - b(S_{i-1}) - 2$.

Moreover: $|S_{i-1}| - 1 \leqslant |S_i| \leqslant |S_{i-1}| + 1$.

So (**) the maximal degree in A of $(-A^2-A^{-2})^{|S_i|-1}$ is at most two more than the one of $(-A^2-A^{-2})^{|S_{i-1}|-1}$.

Putting together (*) and (**) finishes the proof of lemma 11.8.

An easy computation shows that:

$$D_{\mathrm{A}} = c + 2(|\mathrm{A}|-1),$$
$$d_{\mathrm{B}} = -[c+2(|\mathrm{B}|-1)].$$

Proof of theorem 10.1. Let L be any projection of an unsplittable link K in $\mathbf{R}^3$. Then

$$\mathrm{Span}\, f_L = \mathrm{span} <L> \leqslant D_{\mathrm{A}} - d_{\mathrm{B}}$$

and

$$\begin{aligned} D_{\mathrm{A}} - d_{\mathrm{B}} &= c + c + 2|\mathrm{A}| + 2|\mathrm{B}| - 4 \leqslant 2c + 2R-4 \\ &= 2c + 2c + 4-4 = 4c. \end{aligned}$$

As $V_K(t) = f_L(t^{1/4})$, this gives at once a proof of theorem 10.1.

We now proceed towards the proof of theorems 10.2 and 10.3.

LEMMA 11.9. *Let L be a link diagram. Then L is alternating if and only if either all the "A" are shaded or all the "B" are shaded.*

Recall that we suppose that the image of the projection is connected. Recall also that our convention to make a projection alternating was that the "A" should be shaded.

This lemma is essentially Tait's theorem of § 9.

LEMMA 11.10. *Let L be a link diagram, alternating according to the convention. Suppose L without nugatory crossing, i.e. L reduced. Let S be any state, distinct from* A *and* B. *Then*

$$D_S < D_{\mathrm{A}} \quad \text{and} \quad d_{\mathrm{B}} < d_S.$$

Proof of lemma 11.10. The proof begins like the proof of lemma 11.8. We assert that, because the link diagram is reduced, one has

$$D_{S_1} < D_{S_0} = D_{\mathrm{A}}.$$

If the reader goes back to lemma 11.8, he will see that the assertion is all that is needed to get lemma 11.10.

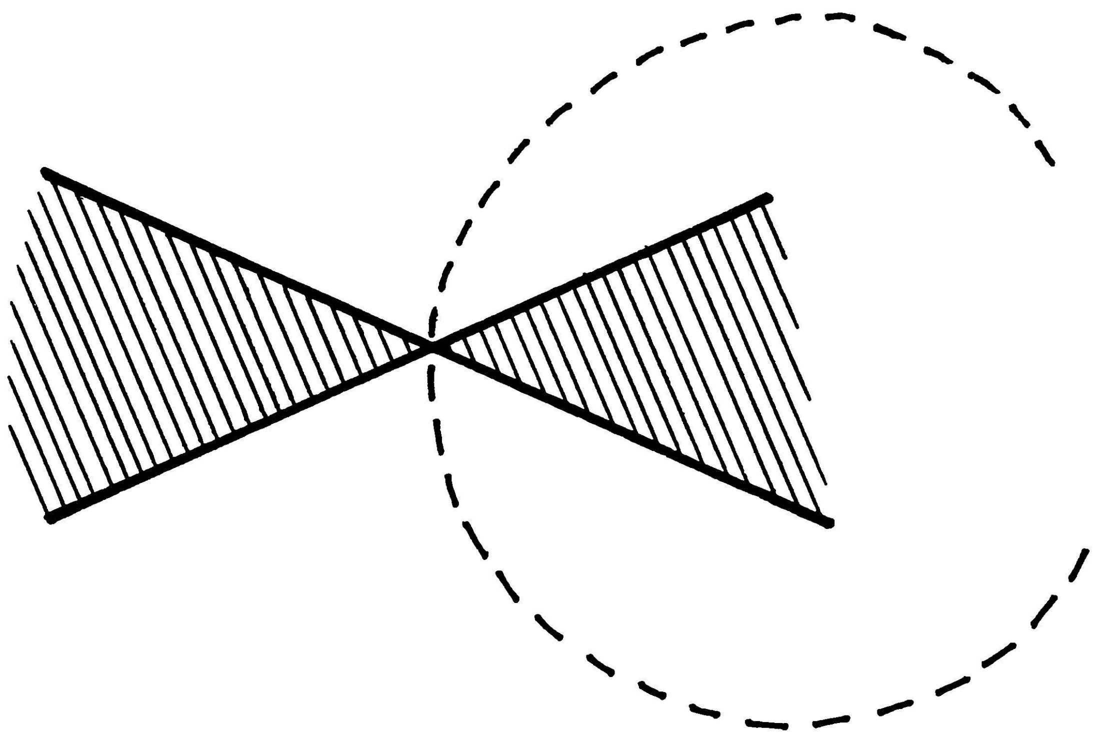
We prove the assertion:

As the link diagram alternates, according the convention the "A" are shaded. So $|\mathrm{A}| = l =$ number of lightened regions.

We claim that $|S_1| = l-1$, the reason being the following: At exactly one double point P of Γ, the marker has passed from A = shade to

B = light. By this operation, two *different* lightened regions have been connected, and the newly shaded surface is still connected. (This immediately implies $|S_1| = l-1$.)

If not, the lightened spots in the neighborhood of P would belong to the same lightened region. One could thus draw a circle entirely in the light, joining the two spots:



This means that L would not be reduced, contrary to the hypotheses. The same kind of argument proves $d_B < d_S$.

This finishes the proof of lemma 11.10.

Notation. Let S be the state obtained by choosing "shade" at every double point and let L be the state obtained by choosing "light" at every double point. Of course, S and L are dual states.

LEMMA 11.11. $|\mathrm{S}| + |\mathrm{L}| = R$.

Proof of lemma 11.11. One has

$$\sigma_S = \Sigma \qquad \lambda_S = \emptyset$$

and

$$\sigma_L = \emptyset \qquad \lambda_L = \Lambda .$$

Then apply the proof of proposition 11.7. Q.E.D.

Proof of theorem 10.2. First of all, we do not restrict the generality by supposing that the diagram alternates according to the convention.

Now lemma 11.10 implies that the highest degree of the monomials in $<L>$ is D_{A} and that the lowest degree is d_{B}. The coefficients of these monomials are different from zero.

Moreover $\mathrm{A} = \mathrm{S}$ and $\mathrm{B} = \mathrm{L}$.

So $|\mathrm{A}| + |\mathrm{B}| = R$ by lemma 11.11.

Hence:

$$\begin{aligned} \text{Span } <L> &= D_{\mathrm{A}} - d_{\mathrm{B}} = 2c + 2|\mathrm{A}| + 2|\mathrm{B}| - 4 = 2c + 2R - 4 \\ &= 2c + 2(c+2) - 4 = 4c \,. \end{aligned}$$

As span $V_K(t) = \frac{1}{4}$ span $<L>$, this finishes the proof.

PROPOSITION 11.12. *Suppose that the graphs* Σ *and* Λ *have no cut vertex. Suppose that for a state* S *we have*

$$|S| + |\check{S}| = R \,.$$

Then $S = \mathrm{S}$ or $S = \mathrm{L}$.

Remark. Σ and Λ have no cut vertex if and only if Γ is not a non-trivial connected sum. See also proof of prop. 11.7.

The proof of proposition 11.12 follows immediately from the consequence of lemma 11.5.

Remark. There is an obvious generalisation of proposition 11.12 to the case of a connected sum. Use the full lemma 11.5 instead of its consequence.

We now state an equivalent form of theorem 10.3.

THEOREM 10.3′. *Let* L *be a link diagram such that* Σ *and* Λ *have no cut vertex. (This will be fulfilled if the link is prime.) Suppose that* span $V_K(t) = c(L)$. *Then* L *is reduced and alternating.*

Remark. There is a generalisation of theorem 10.3′ to the case of a connected sum: the only possible counter-examples to non-alternativity are non-alternating connected sums of alternating links, as in the square knot. We leave this to the reader. (Use generalisation of proposition 11.12.)

Proof of theorem 10.3′. If L were not reduced, we could reduce it. But this would contradict theorem 10.1.

Now, the computation of $D_A - d_B$ in the proof of theorem 10.1 shows that, if span $<L> = 4c$, one has $D_A - d_B = 4c$ and so $|A| + |B| = R$.

As Σ and Λ have no cut vertex, the proposition 11.12 implies that $A = S$ or $A = L$.

By lemma 11.9, this means that L is alternating. Q.E.D.

§ 12. THE PATH FROM VON NEUMANN ALGEBRAS TO KNOT POLYNOMIALS

The discovery of the knot polynomials discussed here is due to Jones' investigations on von Neumann algebras, and not to the flourishing activity in low dimensional topology. In the light of previous work by J. Conway on Alexander's polynomial and of subsequent work by L. Kauffman (among others) on Jones' polynomial, such a genesis may seem unexpected. However this cannot be challenged, and should indeed appear rather as a delight of the subject than as any unpleasant awkwardness. With this point of view, we offer some guidelines for (some of) the surprising relationships put into light by V. Jones' work.

FACTORS OF TYPE II_1

An involution on a complex algebra M is a conjugate linear transformation $x \mapsto x^*$ of M such that $(x^*)^* = x$ and $(xy)^* = y^*x^*$ for all $x, y \in M$. The algebra $L(H)$ of all continuous operators on a Hilbert space H has a canonical involution, with x^* the adjoint of x, defined by $<x^*\xi \mid \eta> = <\xi \mid x\eta>$ for all $\xi, \eta \in H$. A representation of an involutive algebra M on H is a morphism of algebras $\pi: M \to L(H)$ with $\pi(x^*) = (\pi(x))^*$ for all $x \in M$. The algebra $L(H)$ carries several useful topologies, and in particular the weak topology, for which a sequence $(x_i)_{i \in I}$ of operators converges to 0 iff the numerical sequences $(<x_i\xi \mid \eta>)_{i \in I}$ converge to 0 for all pairs (ξ, η) of vectors in H.

A von Neumann algebra is an involutive algebra M with unit which has a faithful representation π on H with $\pi(1) = \mathrm{id}$ and with $\pi(M)$ a weakly closed self-adjoint subalgebra of $L(H)$. (There are several equivalent definitions: see any textbook on the subject, for example one of [Di], [SZ], [Tak].) A von Neumann algebra is defined to be a *factor of type II_1* if

(1) The center of M is reduced to scalar multiples of 1.

(2) There exists a normalized finite trace, namely a linear form $\mathrm{tr}: M \to \mathbf{C}$ with $\mathrm{tr}(1) = 1$ and $\mathrm{tr}(xy) = \mathrm{tr}(yx)$ for all $x, y \in M$.