

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 32 (1986)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** ON THE JONES POLYNOMIAL Swiss Seminar in Berne  
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**Kapitel:** §7. Some properties of  $P_K(l, m)$   
**DOI:** <https://doi.org/10.5169/seals-55091>

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§ 7. SOME PROPERTIES OF  $P_K(l, m)$

In this paragraph we gather some of the basic properties of the polynomial  $P_K(l, m)$ , also denoted  $P(K)$  if the variables are understood.

Let  $K'$  be the oriented link obtained from  $K$  by reversing the orientations of all the components. Then, we have

PROPERTY 7.1.  $P(K') = P(K)$ .

*Proof.* Let  $K_+, K_-, K_0$  be three skein related links. We see that  $K'_+, K'_-$  and  $K'_0$  are also skein related. Hence,

$$lP(K'_+) + l^{-1}P(K'_-) + mP(K'_0) = 0.$$

By uniqueness, this implies  $P(K') = P(K)$  for all  $K$ . (Of course  $\bigcirc' = \bigcirc$ .)

Property 7.1 can also be proved from the definition given in § 6 as follows. If  $K = K(\alpha)$ , then  $K' = K(\alpha')$ , where  $\alpha' = \sigma_{i_r}^{\epsilon_r} \dots \sigma_{i_1}^{\epsilon_1}$  if  $\alpha = \sigma_{i_1}^{\epsilon_1} \dots \sigma_{i_r}^{\epsilon_r}$ . Observe that the operation  $\alpha \mapsto \alpha'$  is a well defined antiautomorphism of  $B_n$ . There is an analogous antiautomorphism of  $H_n$ , sending the monomial  $M = T_{i_1} \dots T_{i_r}$  to  $M' = T_{i_r} \dots T_{i_1}$  and it is easily checked that for all  $x \in H_n$ ,  $\text{Tr}(x) = \text{Tr}(x')$ .

Next, let  $K^\times$  be the mirror image of  $K$ . Then we have

PROPERTY 7.2.  $P_{K^\times}(l, m) = P_K(l^{-1}, m)$ .

*Proof.* Observe that if  $K_+, K_-$ , and  $K_0$  are skein related, then so are  $K_-^\times, K_+^\times$  and  $K_0^\times$  in this order, i.e.

$$lP(K_-^\times) + l^{-1}P(K_+^\times) + mP(K_0^\times) = 0.$$

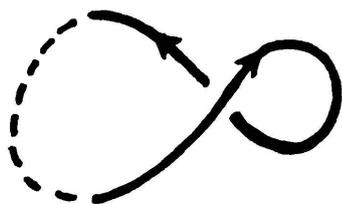
The property follows by uniqueness applied to  $P_{K^\times}(l, m) = P_K(l^{-1}, m)$ .

We shall skip the alternative proof of that property based on braid presentations.

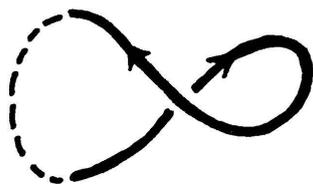
If  $K_1$  and  $K_2$  are two links and  $K_1 \amalg K_2$  their distant union (disjoint, unlinked), then we have

PROPERTY 7.3.  $P(K_1 \amalg K_2) = -\frac{l + l^{-1}}{m} \cdot P(K_1) \cdot P(K_2)$ .

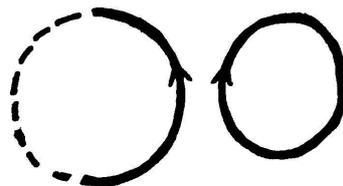
*Proof.* If  $K_2 = \bigcirc$ , this follows from the skein invariance as shown in the following picture



$$L_1 = L_+$$



$$L_1 = L_-$$



$$L_0 = L_1 \amalg \bigcirc$$

which yields

$$lP(K_1) + l^{-1}P(K_1) + mP(K_1 \amalg \bigcirc) = 0,$$

and therefore

$$P(K_1 \amalg \bigcirc) = -\frac{l + l^{-1}}{m} \cdot P(K_1).$$

If  $K_2$  is more complicated, use induction on the complexity of one of its diagrams  $L_2$ . If  $L_2^+, L_2^-, L_2^0$  are skein related, so are  $L_1 \amalg L_2^+, L_1 \amalg L_2^-, L_1 \amalg L_2^0$  for any diagram  $L_1$  of  $K_1$  and Property 7.3 follows.

Second proof. If  $K_1 = K(\alpha)$  with  $\alpha \in B_m$  and  $K_2 = K(\beta)$ , with  $\beta \in B_n$ , then  $K_1 \amalg K_2 = K(\alpha \cdot s(\beta))$  with  $\alpha \cdot s(\beta) \in B_{m+n}$ , where  $s: B_n \rightarrow B_{m+n}$  shifts all indices of the generators  $\sigma_1, \dots, \sigma_{n-1}$  by  $m$ , i.e.  $s(\sigma_i) = \sigma_{m+i}$ . It follows that  $\alpha$  and  $s(\beta)$  commute in  $B_{m+n}$ , and it is easily verified that  $\text{Tr}(\rho(\alpha \cdot s(\beta))) = \text{Tr}(\rho(\alpha)) \cdot \text{Tr}(\rho(\beta))$ . Then,

$$V_{\alpha s(\beta)}(q, z) = (q/zw)^{1/2} \cdot V_\alpha(q, z) \cdot V_\beta(q, z).$$

With  $l = i(z/w)^{1/2}$  and  $m = i(q^{-1/2} - q^{1/2})$ , we have

$$\begin{aligned} -\frac{l + l^{-1}}{m} &= -\frac{(z/w)^{1/2} - (z/w)^{-1/2}}{q^{-1/2} - q^{1/2}} = -\left(\frac{zq}{w}\right)^{1/2} \frac{1 - \frac{w}{z}}{1 - q} \\ &= -\left(\frac{zq}{w}\right)^{1/2} \frac{z - (1 - q + z)}{z(1 - q)} = \left(\frac{q}{zw}\right)^{1/2} \end{aligned}$$

Therefore,

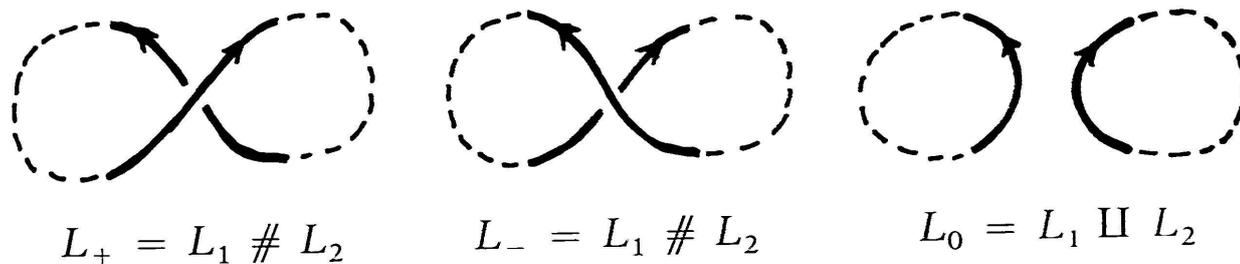
$$V_{\alpha s(\beta)}(q, z) = -\frac{l + l^{-1}}{m} \cdot V_\alpha(q, z) \cdot V_\beta(q, z)$$

as required.

If  $K_1, K_2$  are 2 links, denote by  $K_1 \# K_2$  a connected sum of  $K_1$  and  $K_2$  performed from the unlinked union on any choice of components.

PROPERTY 7.4.  $P(K_1 \# K_2) = P(K_1) \cdot P(K_2)$ .

*Proof.* We use the skein relation



where  $L_1$  and  $L_2$  are diagrams of  $K_1$  and  $K_2$ .

This gives the formula

$$lP(L_1 \# L_2) + l^{-1}P(L_1 \# L_2) + mP(L_1 \amalg L_2) = 0.$$

Solving for  $P(L_1 \# L_2)$  and using property 7.3, the factor  $-(l+l^{-1})/m$  cancels out and the result follows.

The proof using braid presentations is more complicated and will be omitted.

Since  $P: \mathcal{L} \rightarrow \mathbb{Z}[l, l^{-1}, m, m^{-1}]$  is the universal skein invariant, it must specialize to the Alexander polynomial and to the one-variable Jones polynomial.

Specifically, define

$$\Delta_K(t) = P_K(i, i(t^{1/2} - t^{-1/2})),$$

then we have

PROPERTY 7.5.  $\Delta_K(t)$  satisfies the skein invariance

- (1)  $\Delta_{\bigcirc}(t) = 1,$
- (2)  $\Delta(K_+) - \Delta(K_-) + (t^{1/2} - t^{-1/2})\Delta(K_0) = 0,$

which characterizes the Alexander polynomial as normalized by J. Conway.

(See L. Kauffman, [Ka<sub>1</sub>].)

Recall from § 3 that the exponent of  $m$  in each monomial of  $P_K(l, m)$  is congruent mod 2 to  $r(K) - 1$ , where  $r(K)$  is the number of components

of  $K$ . Hence, for a knot, a link with a single component, the exponent of  $m$  in  $P_K(l, m)$  is even and therefore  $\Delta_K(t) = P_K(i, i(t^{1/2} - t^{-1/2}))$  is indeed a Laurent polynomial in  $t$ .

To obtain the one-variable Jones polynomial we use the substitution  $l = it, m = i(t^{1/2} - t^{-1/2})$ . Explicitly,

$$V_K(t) = P_K(it, i(t^{1/2} - t^{-1/2}))$$

Then we have

PROPERTY 7.6.  $V_K(t)$  satisfies the skein invariance

$$tV(K_+) - t^{-1}V(K_-) + (t^{1/2} - t^{-1/2})V(K_0) = 0,$$

which (together with  $V(\bigcirc) = 1$ ) characterizes Jones one-variable polynomial, with the sign conventions used in reference [Jo<sub>3</sub>].

Whereas  $P_K(l, m)$  determines  $\Delta_K(t)$  and  $V_K(t)$ , it is known that there are no other relations between these polynomials. More precisely:

(1) The Alexander polynomial  $\Delta_K(t)$  does not determine Jones polynomial  $V_K(t)$  because the trivial knot  $\bigcirc$  and Conway's eleven crossing knot  $11_{471}$  have  $\Delta(t) = 1$ , but  $V_K(t) \neq 1$  for  $K = 11_{471}$ .

(2)  $V_K(t)$  does not determine  $\Delta_K(t)$ : The knots  $4_1$  and  $11_{388}$  have the same  $V(t)$  but different  $\Delta(t)$ .

(3)  $V_K(t)$  and  $\Delta_K(t)$  together do not determine  $P_K(l, m)$ : The knot  $11_{388}$  and its mirror image have the same  $V(t)$  and  $\Delta(t)$  but different  $P(l, m)$ .

For more details on these questions, see [L.-M.].

We now turn to L. Kauffman's definition of the one-variable Jones polynomial  $V_K(t)$  directly from the link diagram.

## § 8. L. KAUFFMAN'S APPROACH TO V. JONES' ONE-VARIABLE POLYNOMIAL

The importance of Kauffman's approach [Ka<sub>3</sub>] is that it gives a new way to define and compute Jones polynomial  $V_K(t)$ . It is by using this definition that Kauffman and Murasugi prove their theorems about alternating links (see § 10 and 11).

Let  $L$  be an *unoriented* link diagram. Look at a double point; with no string orientation, they all look the same, up to a local homeomorphism: