Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 32 (1986)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: ON THE JONES POLYNOMIAL Swiss Seminar in Berne
Autor: de la Harpe, Pierre / Kervaire, Michel / Weber, Claude
Kapitel: §7. Some properties of $P_K(l, m)$
DOI: https://doi.org/10.5169/seals-55091

Nutzungsbedingungen
Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Mehr erfahren

Conditions d'utilisation
L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. En savoir plus

Terms of use
The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. Find out more

Download PDF: 07.01.2026

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

§ 7. Some properties of $P_K(l, m)$

In this paragraph we gather some of the basic properties of the polynomial $P_K(l, m)$, also denoted $P(K)$ if the variables are understood.

Let K' be the oriented link obtained from K by reversing the orientations of all the components. Then, we have

Property 7.1. $P(K') = P(K)$.

Proof. Let K_+, K_-, K_0 be three skein related links. We see that K'_+, K'_- and K'_0 are also skein related. Hence,

$$lP(K'_+) + l^{-1}P(K'_-) + mP(K'_0) = 0 .$$

By uniqueness, this implies $P(K') = P(K)$ for all K. (Of course $\bigcirc' = \bigcirc$.)

Property 7.1 can also be proved from the definition given in § 6 as follows. If $K = K(\alpha)$, then $K' = K(\alpha')$, where $\alpha' = \sigma_{i_r}^{\varepsilon_r} \dots \sigma_{i_1}^{\varepsilon_1}$ if $\alpha = \sigma_{i_1}^{\varepsilon_1} \dots \sigma_{i_r}^{\varepsilon_r}$. Observe that the operation $\alpha \mapsto \alpha'$ is a well defined antiautomorphism of B_n. There is an analogous antiautomorphism of H_n, sending the monomial $M = T_{i_1} \dots T_{i_r}$ to $M' = T_{i_r} \dots T_{i_1}$ and it is easily checked that for all $x \in H_n$, $\mathrm{Tr}(x) = \mathrm{Tr}(x')$.

Next, let $K^\times$ be the mirror image of K. Then we have

Property 7.2. $P_{K^\times}(l, m) = P_K(l^{-1}, m)$.

Proof. Observe that if K_+, K_-, and K_0 are skein related, then so are $K_-^\times$, $K_+^\times$ and $K_0^\times$ *in this order*, i.e.

$$lP(K_-^\times) + l^{-1}P(K_+^\times) + mP(K_0^\times) = 0 .$$

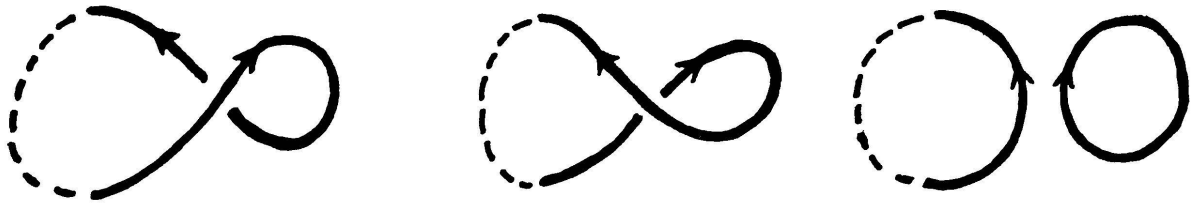
The property follows by uniqueness applied to $P_{K^\times}(l, m) = P_K(l^{-1}, m)$.

We shall skip the alternative proof of that property based on braid presentations.

If K_1 and K_2 are two links and $K_1 \amalg K_2$ their distant union (disjoint, unlinked), then we have

Property 7.3. $P(K_1 \amalg K_2) = -\dfrac{l + l^{-1}}{m} . P(K_1) . P(K_2)$.

Proof. If $K_2 = \bigcirc$, this follows from the skein invariance as shown in the following picture



$L_1 = L_+$ $\qquad$ $L_1 = L_-$ $\qquad$ $L_0 = L_1 \amalg \bigcirc$

which yields

$$lP(K_1) + l^{-1}P(K_1) + mP(K_1 \amalg \bigcirc) = 0\,,$$

and therefore

$$P(K_1 \amalg \bigcirc) = -\frac{l + l^{-1}}{m} \,.\, P(K_1)\,.$$

If K_2 is more complicated, use induction on the complexity of one of its diagrams L_2. If L_2^+, L_2^-, L_2^0 are skein related, so are $L_1 \amalg L_2^+$, $L_1 \amalg L_2^-$, $L_1 \amalg L_2^0$ for any diagram L_1 of K_1 and Property 7.3 follows.

Second proof. If $K_1 = K(\alpha)$ with $\alpha \in B_m$ and $K_2 = K(\beta)$, with $\beta \in B_n$, then $K_1 \amalg K_2 = K(\alpha \,.\, s(\beta))$ with $\alpha \,.\, s(\beta) \in B_{m+n}$, where $s\colon B_n \to B_{m+n}$ shifts all indices of the generators $\sigma_1, \ldots, \sigma_{n-1}$ by m, i.e. $s(\sigma_i) = \sigma_{m+i}$. It follows that α and $s(\beta)$ commute in B_{m+n}, and it is easily verified that $\mathrm{Tr}(\rho(\alpha \,.\, s(\beta))) = \mathrm{Tr}(\rho(\alpha)) \,.\, \mathrm{Tr}(\rho(\beta))$. Then,

$$V_{\alpha s(\beta)}(q, z) = (q/zw)^{1/2} \,.\, V_\alpha(q, z) \,.\, V_\beta(q, z)\,.$$

With $l = i(z/w)^{1/2}$ and $m = i(q^{-1/2} - q^{1/2})$, we have

$$-\frac{l + l^{-1}}{m} = -\frac{(z/w)^{1/2} - (z/w)^{-1/2}}{q^{-1/2} - q^{1/2}} = -\left(\frac{zq}{w}\right)^{1/2} \frac{1 - \dfrac{w}{z}}{1 - q}$$
$$= -\left(\frac{zq}{w}\right)^{1/2} \frac{z - (1 - q + z)}{z(1 - q)} = \left(\frac{q}{zw}\right)^{1/2}$$

Therefore,

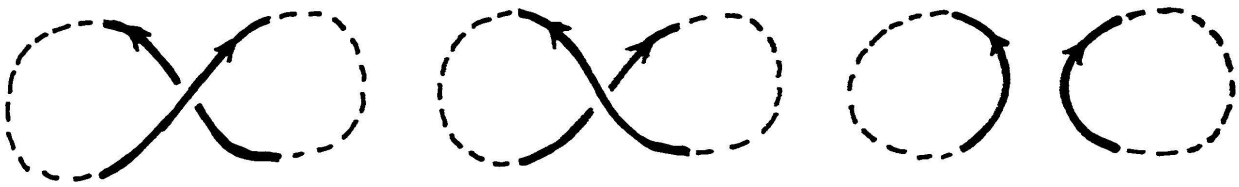
$$V_{\alpha s(\beta)}(q, z) = -\frac{l + l^{-1}}{m} \,.\, V_\alpha(q, z) \,.\, V_\beta(q, z)$$

as required.

If K_1, K_2 are 2 links, denote by $K_1 \# K_2$ a connected sum of K_1 and K_2 performed from the unlinked union on any choice of components.

PROPERTY 7.4. $P(K_1 \# K_2) = P(K_1) \cdot P(K_2)$.

Proof. We use the skein relation



$L_+ = L_1 \# L_2$ $\qquad$ $L_- = L_1 \# L_2$ $\qquad$ $L_0 = L_1 \sqcup L_2$

where L_1 and L_2 are diagrams of K_1 and K_2.

This gives the formula

$$lP(L_1 \# L_2) + l^{-1}P(L_1 \# L_2) + mP(\overline{L_1 \sqcup L_2}) = 0 .$$

Solving for $P(L_1 \# L_2)$ and using property 7.3, the factor $-(l+l^{-1})/m$ cancels out and the result follows.

The proof using braid presentations is more complicated and will be omitted.

Since $P: \mathscr{L} \to \mathbf{Z}[l, l^{-1}, m, m^{-1}]$ is the universal skein invariant, it must specialize to the Alexander polynomial and to the one-variable Jones polynomial.

Specifically, define

$$\Delta_K(t) = P_K(i, i(t^{1/2} - t^{-1/2})) ,$$

then we have

PROPERTY 7.5. $\Delta_K(t)$ *satisfies the skein invariance*

(1) $\Delta_{\bigcirc}(t) = 1$,

(2) $\Delta(K_+) - \Delta(K_-) + (t^{1/2} - t^{-1/2})\Delta(K_0) = 0$,

which characterizes the Alexander *polynomial as normalized by* J. Conway. (See L. Kauffman, [Ka$_1$].)

Recall from § 3 that the exponent of m in each monomial of $P_K(l, m)$ is congruent mod 2 to $r(K) - 1$, where $r(K)$ is the number of components

of K. Hence, for a knot, a link with a single component, the exponent of m in $P_K(l, m)$ is even and therefore $\Delta_K(t) = P_K(i, i(t^{1/2}-t^{-1/2}))$ is indeed a Laurent polynomial in t.

To obtain the one-variable Jones polynomial we use the substitution $l = it$, $m = i(t^{1/2}-t^{-1/2})$. Explicitly,

$$V_K(t) = P_K(it, i(t^{1/2}-t^{-1/2}))$$

Then we have

PROPERTY 7.6. *$V_K(t)$ satisfies the skein invariance*

$$tV(K_+) - t^{-1}V(K_-) + (t^{1/2}-t^{-1/2})V(K_0) = 0\,,$$

which (together with $V(\bigcirc)=1$) characterizes Jones *one-variable polynomial, with the sign conventions used in reference* [Jo$_3$].

Whereas $P_K(l, m)$ determines $\Delta_K(t)$ and $V_K(t)$, it is known that there are no other relations between these polynomials. More precisely:

(1) The Alexander polynomial $\Delta_K(t)$ does not determine Jones polynomial $V_K(t)$ because the trivial knot $\bigcirc$ and Conway's eleven crossing knot 11_{471} have $\Delta(t) = 1$, but $V_K(t) \neq 1$ for $K = 11_{471}$.

(2) $V_K(t)$ does not determine $\Delta_K(t)$: The knots 4_1 and 11_{388} have the same $V(t)$ but different $\Delta(t)$.

(3) $V_K(t)$ and $\Delta_K(t)$ together do not determine $P_K(l, m)$: The knot 11_{388} and its mirror image have the same $V(t)$ and $\Delta(t)$ but different $P(l, m)$.

For more details on these questions, see [L.-M.].

We now turn to L. Kauffman's definition of the one-variable Jones polynomial $V_K(t)$ directly from the link diagram.

§ 8. L. KAUFFMAN'S APPROACH TO V. JONES' ONE-VARIABLE POLYNOMIAL

The importance of Kauffman's approach [Ka$_3$] is that it gives a new way to define and compute Jones polynomial $V_K(t)$. It is by using this definition that Kauffman and Murasugi prove their theorems about alternating links (see § 10 and 11).

Let L be an *unoriented* link diagram. Look at a double point; with no string orientation, they all look the same, up to a local homeomorphism: