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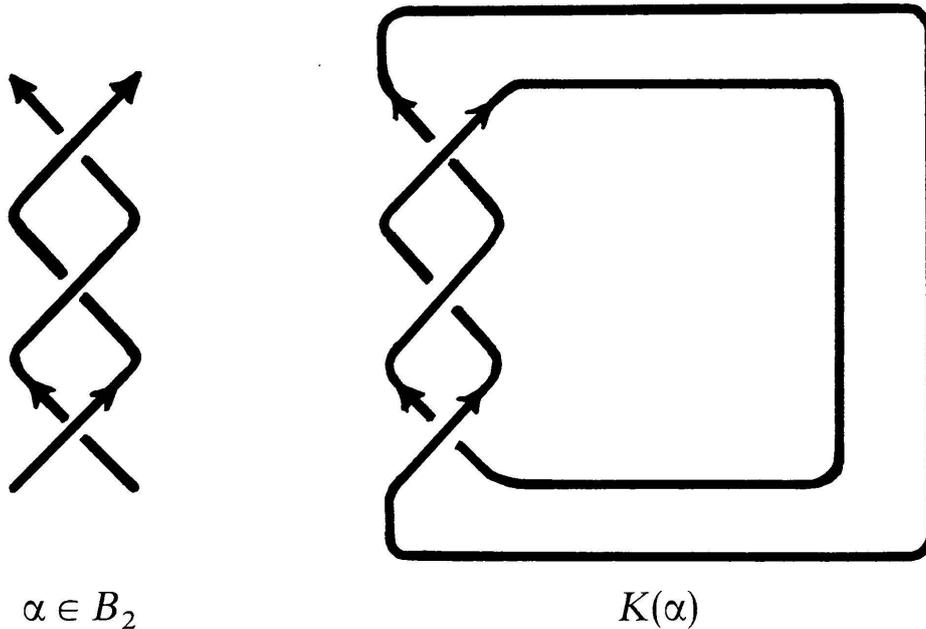
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§ 6. EXISTENCE OF THE TWO-VARIABLE POLYNOMIAL

The polynomial will be defined as a braid invariant.

A braid α gives rise to a link $K(\alpha)$ by the "closing" operation, as shown in the picture



Closing a braid

Recall that every oriented link is ambient isotopic to a closed braid, as was already known to Alexander [Al]. (See also [Mo].)

Now, let $K = \mathbf{C}(q, z)$ be the rational field in 2 variables q, z over the complex numbers and let $w = 1 - q + z \in K$.

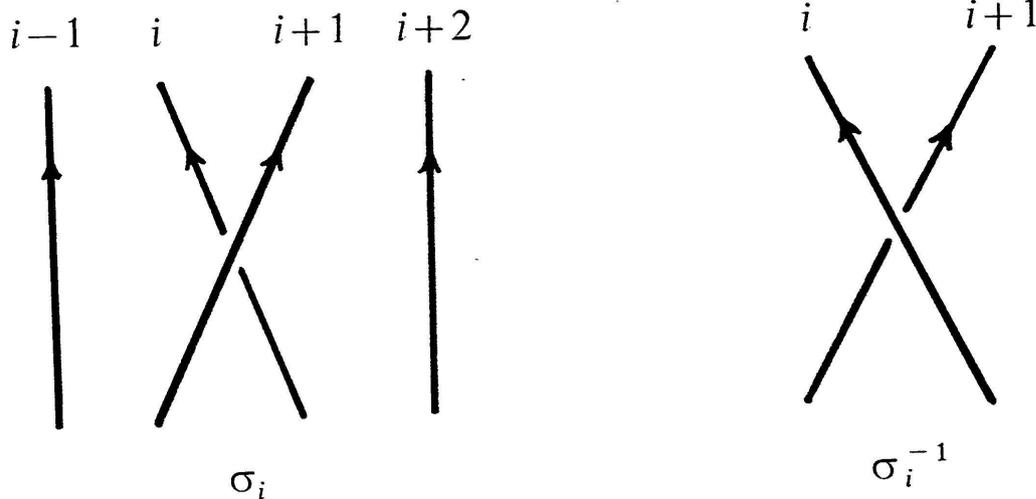
With every braid $\alpha \in B_n$ we will associate an element $V_\alpha(q, z)$ in the quadratic extension $K(\sqrt{q/zw})$ of K .

It is a quite remarkable fact that $V_\alpha(q, z)$ will depend only on the link $K(\alpha)$ obtained from α by closing the braid as shown above, and not on α itself.

Thus, we will be able to define $V_K(q, z) = V_\alpha(q, z)$, where α is any braid such that K is ambient isotopic to $K(\alpha)$.

In order to define $V_\alpha(q, z)$ we now proceed to fix some notations.

We use the following conventions regarding the generators $\sigma_1, \dots, \sigma_{n-1} \in B_n$ of the braid group on n strings B_n .



Recall that B_n has a presentation on the generators $\sigma_1, \dots, \sigma_{n-1}$ with relations

$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{if } |i - j| \geq 2,$$

and

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$

for $i = 1, \dots, n-2$.

Note that there is a well defined homomorphism $e: B_n \rightarrow \mathbf{Z}$ given by $e(\sigma_i) = 1, i = 1, \dots, n-1$, on the generators. We call e the exponent sum.

There is also an obvious representation $\rho: B_n \rightarrow H_n$ determined by

$$\rho(\sigma_i) = T_i.$$

Note that $T_i \in H_n$ is invertible in H_n : $T_i^{-1} = \frac{1}{q}(1 - q + T_i)$. Now, let $\alpha \in B_n$.

The corresponding element $V_\alpha(q, z) \in K(\sqrt{q/zw})$ is defined by the formula

$$V_\alpha(q, z) = (1/z)^{(n+e(\alpha)-1)/2} \cdot (q/w)^{(n-e(\alpha)-1)/2} \cdot \text{Tr}(\rho(\alpha)),$$

where $w = 1 - q + z$, $\rho: B_n \rightarrow H_n$ is as above, and $e(\alpha)$ is the exponent sum of α .

In order to show that $V_\alpha(q, z)$ depends only on the link $K(\alpha)$, we appeal to Markov's theorem which gives necessary and sufficient conditions for 2 braids $\alpha \in B_m, \beta \in B_n$ to produce isotopic links $K(\alpha), K(\beta)$ by closing.

Define a Markov move of type 1 to be the operation of replacing a braid $\alpha \in B_n$ by a conjugate $\gamma \alpha \gamma^{-1} \in B_n$ with $\gamma \in B_n$.

A Markov move of type 2 consists in replacing $\alpha \in B_n$ by $\alpha \cdot \sigma_n$ or $\alpha \cdot \sigma_n^{-1}$ in B_{n+1} . Or, replacing $\alpha \cdot \sigma_n \in B_{n+1}$, resp. $\alpha \cdot \sigma_n^{-1} \in B_{n+1}$ by $\alpha \in B_n$ if α is a word in the generators $\sigma_1, \dots, \sigma_{n-1}$ only.

THEOREM (Markov). *Let $\alpha \in B_m$, $\beta \in B_n$ be two braids. Then, $K(\alpha)$ and $K(\beta)$ are ambient isotopic as oriented links iff there exists a finite sequence of Markov moves carrying α to β .*

For a proof, see [Mo].

Thus, we have to show that $V_\alpha(q, z)$ is unchanged by Markov moves on α .

Let $\alpha \in B_n$, $\gamma \in B_n$ and $\beta = \gamma\alpha\gamma^{-1}$. Then the string numbers of α and β are the same. Also $e(\alpha) = e(\beta)$, and $\text{Tr}(\rho(\beta)) = \text{Tr}(\rho(\gamma)\rho(\alpha)\rho(\gamma)^{-1}) = \text{Tr}(\rho(\alpha))$. Hence, $V_\beta(q, z) = V_\alpha(q, z)$.

If $\alpha \in B_n$ and $\beta = \alpha \cdot \sigma_n \in B_{n+1}$, we have $e(\beta) = e(\alpha) + 1$, $n(\beta) = n + 1$ (where $n = n(\alpha)$). Thus,

$$\begin{aligned} V_\beta(q, z) &= (1/z)^{(n+e(\alpha)-1)/2} \cdot (q/w)^{(n-e(\alpha)-1)/2} \cdot (1/z) \cdot \text{Tr}(\rho(\alpha) \cdot T_n) \\ &= V_\alpha(q, z), \text{ as desired.} \end{aligned}$$

If $\alpha \in B_n$ and $\beta = \alpha\sigma_n^{-1} \in B_{n+1}$, then $e(\beta) = e(\alpha) - 1$, $n(\beta) = n + 1$ and

$$V_\beta(q, z) = (1/z)^{(n+e(\alpha)-1)/2} \cdot (q/w)^{(n-e(\alpha)-1)/2} \cdot (q/w) \cdot \text{Tr}\{\rho(\alpha) \cdot T_n^{-1}\}.$$

Now,

$$\rho(\alpha) \cdot T_n^{-1} = \frac{1}{q} \rho(\alpha) \cdot (1 - q + T_n),$$

and

$$\text{Tr}\{\rho(\alpha) \cdot T_n^{-1}\} = \frac{1}{q} (1 - q + z) \cdot \text{Tr}(\rho(\alpha)) = (w/q) \cdot \text{Tr}(\rho(\alpha)).$$

Hence, again $V_\beta(q, z) = V_\alpha(q, z)$.

Thus $V_\alpha(q, z)$ is well defined as an invariant of oriented links.

On the face of it, $V_\alpha(q, z)$ does not look much so far like a polynomial with integral coefficients. However, as it turns out, a slick change of variables will do the trick. We have:

PROPOSITION 6.1. *There exists for each link K , a unique Laurent polynomial $P_K(l, m) \in \mathbf{Z}[l, l^{-1}, m, m^{-1}]$, such that*

$$P_K(i(z/w)^{1/2}, i(q^{-1/2} - q^{1/2})) = V_\alpha(q, z),$$

whenever α is a braid giving rise (up to ambient isotopy, of course) to the link K by closing.

Thus, $V_\alpha(q, z)$ becomes the Laurent polynomial $P_{K(\alpha)}(l, m)$ in

$$\mathbf{Z}[l, l^{-1}, m, m^{-1}]$$

after the change of variables

$$l = i(z/w)^{1/2}, \quad m = i(q^{-1/2} - q^{1/2}).$$

The key to the proof of this fact is the skein invariance of $V_\alpha(q, z)$ which we now proceed to show.

Let $\beta, \gamma \in B_n$ be two braids and let $\alpha_+, \alpha_-, \alpha_0$ be the three braids

$$\alpha_+ = \beta\sigma_k\gamma, \quad \alpha_- = \beta\sigma_k^{-1}\gamma, \quad \alpha_0 = \beta\gamma,$$

for some index $k \leq n-1$.

For any braid $\alpha \in B_n$, with exponent sum $e = e(\alpha)$, define

$$W_\alpha(q, z) = (1/z)^{(n+e-1)/2} \cdot (q/w)^{(n-e-1)/2} \cdot \rho(\alpha) \in H_n,$$

where H_n is now the Hecke algebra over $K(\sqrt{q/zw})$ with $K = \mathbf{C}(q, z)$, corresponding to q .

SKEIN INVARIANCE LEMMA. Set $l = i(z/w)^{1/2}$ and $m = i(q^{-1/2} - q^{1/2})$. Then, we have the relation

$$lW_{\alpha_+} + l^{-1}W_{\alpha_-} + mW_{\alpha_0} = 0.$$

Taking the trace, we obtain from this lemma the skein invariance of $V_\alpha(q, z)$: With the same notations as above

$$lV_{\alpha_+} + l^{-1}V_{\alpha_-} + mV_{\alpha_0} = 0.$$

Proof of the lemma. Set $e = e(\alpha_0)$, and observe that we have

$$W_{\alpha_+} = (1/z)^{1/2}(q/w)^{-1/2}(1/z)^{(n+e-1)/2}(q/w)^{(n-e-1)/2} \cdot \rho(\beta)T_k\rho(\gamma),$$

$$W_{\alpha_-} = (1/z)^{-1/2}(q/w)^{1/2}(1/z)^{(n+e-1)/2}(q/w)^{(n-e-1)/2} \cdot \rho(\beta)T_k^{-1}\rho(\gamma).$$

An easy calculation now gives for $lW_{\alpha_+} + l^{-1}W_{\alpha_-} + mW_{\alpha_0}$ an expression of the form

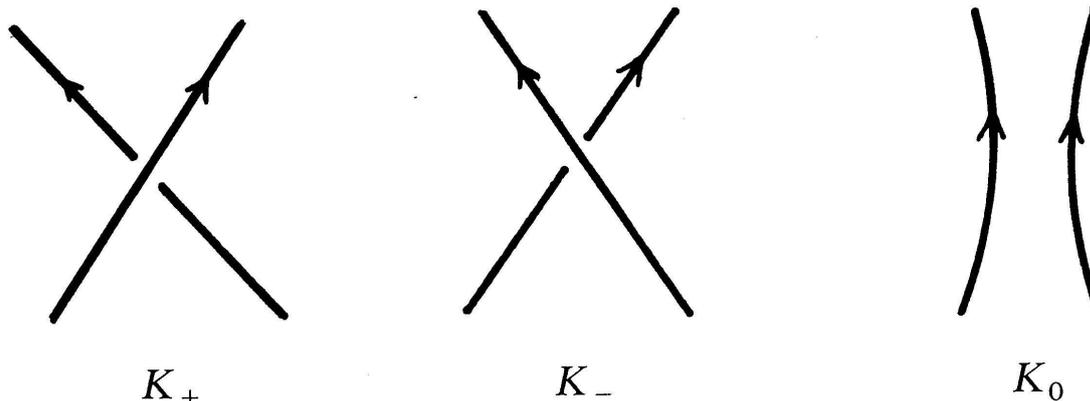
$$i(1/z)^{(n+e-1)/2}(q/w)^{(n-e-1)/2} \cdot \rho(\beta) \cdot C \cdot \rho(\gamma),$$

where

$$C = q^{-1/2}T_k - q^{1/2}T_k^{-1} + q^{-1/2} - q^{1/2}.$$

Recalling that $T_k^{-1} = q^{-1}(1-q+T_k)$, it is easy to verify that $C = 0$.

Proof of proposition 6.1. Let K_+ , K_- and K_0 be three skein related links.



It is an obvious consequence of the classical proof of Alexander's theorem that the three links can be presented as closed braids of the form

$$K_+ = K(\alpha_+), \quad K_- = K(\alpha_-), \quad K_0 = K(\alpha_0),$$

with $\alpha_+ = \beta\sigma_k\gamma$, $\alpha_- = \beta\sigma_k^{-1}\gamma$, $\alpha_0 = \beta\gamma$ for some braids $\beta, \gamma \in B_n$ and some index $k \leq n-1$.

Writing $V(K)$ for V_α if $K = K(\alpha)$, it follows from the skein invariance lemma above, that

$$lV(K_+) + l^{-1}V(K_-) + mV(K_0) = 0, \quad .$$

if K_+ , K_- and K_0 are skein related.

It follows now by induction on the link complexity, as in the proof of uniqueness in § 3, that $V(K)$ is actually a Laurent polynomial with integer coefficients in the variables l and m .

We change notation and set $P_K(l, m) \in \mathbf{Z}[l, l^{-1}, m, m^{-1}]$, where

$$P_{K(\alpha)}(i(z/w)^{1/2}, i(q^{-1/2} - q^{1/2})) = V_\alpha(q, z)$$

Since it is obvious that $P_K(l, m) = 1$ if K is the unknot \bigcirc , we have shown that $P: \mathcal{L} \rightarrow \mathbf{Z}[l, l^{-1}, m, m^{-1}]$ exists as a skein invariant. It is universal by what we saw before in § 3.