

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 32 (1986)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** ON THE JONES POLYNOMIAL Swiss Seminar in Berne  
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**Kapitel:** §5. The trace  
**DOI:** <https://doi.org/10.5169/seals-55091>

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## § 5. THE TRACE

The fundamental idea of V. Jones which led him to the definition of his original one-variable polynomial is the construction of the trace. Originally, V. Jones used algebras which are quotients of the algebras  $H_n$ . The lifting of the trace to the Hecke algebras  $H_n$  was observed by A. Ocneanu.

The trace will commute with the inclusion  $H_n \rightarrow H_{n+1}$  and therefore yield a trace on the direct limit of the  $H_n$ 's. (Compare with the discussion in § 12.)

**THEOREM.** *Let  $K$  be a field and let  $q, z \in K$  be two elements of  $K$ . Let  $H_n$  be the Hecke algebra over  $K$  corresponding to  $q$ . There exists a trace  $\text{Tr}: H_n \rightarrow K$  compatible with the inclusion  $H_n \rightarrow H_{n+1}$ , i.e. the diagram*

$$\begin{array}{ccc} H_n & \xrightarrow{\quad} & H_{n+1} \\ & \searrow \text{Tr} & \swarrow \text{Tr} \\ & K & \end{array}$$

*commutes, and such that*

- (1)  $\text{Tr}(1) = 1$ ,
- (2)  $\text{Tr}$  is  $K$ -linear and  $\text{Tr}(ab) = \text{Tr}(ba)$ ,
- (3) If  $a, b \in H_n$ , then  $\text{Tr}(aT_n b) = z\text{Tr}(ab)$ .

Notice that the last property enables us to calculate  $\text{Tr}(x)$  for an arbitrary  $x \in H_n$  by using the fact that monomials in normal form generate  $H_n$  over  $K$ . For instance,

$$\begin{aligned} \text{Tr}(T_1) &= z, \\ \text{Tr}(T_1 T_2) &= \text{Tr}(T_2 T_1) = z^2, \\ \text{Tr}(T_1 T_2 T_1) &= z\text{Tr}(T_1^2) = z((q-1)z + q). \end{aligned}$$

*Proof.* The  $K$ -linear map  $\text{Tr}: H_{n+1} \rightarrow K$  is defined by induction on  $n$ , using the structure lemma of § 4 (Proposition 4.1):

$$\varphi: H_n \oplus H_n \otimes_{H_{n-1}} H_n \xrightarrow{\sim} H_{n+1}.$$

Starting with  $\text{Tr}: H_0 = K \rightarrow K$  the identity, one defines  $\text{Tr}: H_{n+1} \rightarrow K$  by  $\text{Tr}(x) = \text{Tr}(a) + \sum_i z\text{Tr}(b_i c_i)$ , if  $\varphi(a + \sum_i b_i \otimes c_i) = x$ .

It is clear that if  $a, b \in H_n$ , then

$$\text{Tr}(aT_n b) = z\text{Tr}(ab),$$

since  $\varphi(a \otimes b) = aT_n b$ .

The only statement to be proved is then:

$$\text{Tr}(xy) = \text{Tr}(yx) \quad \text{for all } x, y \in H_{n+1}.$$

This is proved by induction on  $n$ .

We may assume that  $x$  and  $y$  are monomials containing  $T_n$  at most once.

If  $y$  does not contain  $T_n$  at all, then writing  $x = x'T_n x''$ , where  $x', x''$  are monomials in  $T_1, \dots, T_{n-1}$ , one has

$$\text{Tr}(xy) = z\text{Tr}(x'x''y) = z\text{Tr}(yx'x'') = \text{Tr}(yx'T_n x'') = \text{Tr}(yx).$$

If  $y$  contains  $T_n$ , it suffices to check the case where  $x = aT_n b$  and  $y = T_n$ , as is easily verified. (Here  $a, b \in H_n$ .)

There are various cases depending on whether or not the elements  $a$  and  $b$  actually contain  $T_{n-1}$ . The worst case is the one in which  $a = a'T_{n-1}a'', b = b'T_{n-1}b''$  with  $a', a'', b', b''$  belonging to  $H_{n-1}$ . We have then

$$\begin{aligned}\text{Tr}(aT_n b T_n) &= z((q-1)\text{Tr}(ab) + q\text{Tr}(ab'b'')) \\ \text{Tr}(T_n a T_n b) &= z((q-1)\text{Tr}(ab) + q\text{Tr}(a'a''b)).\end{aligned}$$

But

$$\text{Tr}(ab'b'') = \text{Tr}(a'T_{n-1}a''b'b'') = z\text{Tr}(a'a''b'b''),$$

and

$$\text{Tr}(a'a''b) = \text{Tr}(a'a''b'T_{n-1}b'') = z\text{Tr}(a'a''b'b'').$$

Hence,

$$\text{Tr}(aT_n b T_n) = \text{Tr}(T_n a T_n b)$$

as desired.