

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 32 (1986)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: ON THE JONES POLYNOMIAL Swiss Seminar in Berne
Autor: de la Harpe, Pierre / Kervaire, Michel / Weber, Claude
Kapitel: §4. Hecke algebras
DOI: <https://doi.org/10.5169/seals-55091>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 10.12.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

§ 4. HECKE ALGEBRAS

In this section we isolate the classical facts about Hecke algebras which we will need in the next two sections in order to prove the existence of P . The knowledgeable reader can thus skip this paragraph and proceed directly to § 5.

Let K be a field and let $q \in K$ be some element of K .

The Hecke algebra H_n over K corresponding to q is the associative K -algebra with unit 1, generated by T_1, \dots, T_{n-1} subject to the following relations

$$T_i T_j = T_j T_i \quad \text{whenever} \quad |i-j| \geq 2,$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \quad \text{and}$$

$$T_i^2 = (q-1)T_i + q$$

for all $i, j \in \{1, \dots, n-1\}$, with of course $i \leq n-2$ for the second family of relations.

We see that there is a natural map $H_n \rightarrow H_{n+1}$ of K -algebras which make H_{n+1} a (H_n, H_n) -bimodule. We think of $q \in K$ as being fixed once and for all.

Consider also the (H_n, H_n) -bimodule $H_n \oplus H_n \otimes_{H_{n-1}} H_n$.

PROPOSITION 4.1. *There is a natural map of (H_n, H_n) -bimodules*

$$\varphi: H_n \oplus H_n \otimes_{H_{n-1}} H_n \rightarrow H_{n+1}$$

given by $\varphi(a + \sum_i b_i \otimes c_i) = a + \sum_i b_i T_n c_i$.

Moreover, φ is an isomorphism.

The proof of this proposition will occupy the remainder of this section. We have divided it into seven claims.

CLAIM 1. *The map φ is well defined.*

Proof. If $u \in H_{n-1}$, then

$$\varphi(bu \otimes c) = bu T_n c, \quad \text{and} \quad \varphi(b \otimes uc) = b T_n uc.$$

But u is a K -linear combination of monomials in T_1, \dots, T_{n-2} which commute with T_n in H_{n+1} . Hence, $bu T_n c = b T_n uc$, and so φ is well defined.

CLAIM 2. *The map φ is surjective.*

We have to show that H_{n+1} is generated as a vector space over K by the monomials with at most one occurrence of T_n .

The proof will be by induction on n . Let M be a monomial in T_1, \dots, T_n with two occurrences of T_n at least. Displaying two consecutive occurrences of T_n in M , we write $M = M_1 T_n M_2 T_n M_3$, where we can assume that M_2 is a monomial in T_1, \dots, T_{n-1} only. Assume by induction that M_2 contains T_{n-1} at most once. If M_2 does not contain T_{n-1} at all, then

$$M = M_1 M_2 T_n^2 M_3 = (q-1)M_1 M_2 T_n M_3 + qM_1 M_2 M_3,$$

reducing the number of occurrences of T_n in each new monomial. If M_2 contains T_{n-1} exactly once, we can write $M_2 = M' T_{n-1} M''$, with M', M'' monomials in T_1, \dots, T_{n-2} and then,

$$M = M_1 M' T_n T_{n-1} T_n M'' M_3,$$

using the fact that T_1, \dots, T_{n-2} commute with T_n . But now, $T_n T_{n-1} T_n = T_{n-1} T_n T_{n-1}$ yields

$$M = M_1 M' T_{n-1} T_n T_{n-1} M'' M_3,$$

reducing again the number of occurrences of T_n .

Hence, every element of H_{n+1} is a sum $a + \sum_i b_i T_n c_i$ with a, b_i, c_i coming from H_n and it is now clear that φ is surjective.

CLAIM 3. *Monomials in normal form generate H_{n+1} over K .*

We have actually proved a little more than was stated in Claim 2. Consider the following lists of monomials:

$$S_1 = \{1, T_1\},$$

$$S_2 = \{1, T_2, T_2 T_1\},$$

$$S_3 = \{1, T_3, T_3 T_2, T_3 T_2 T_1\},$$

...

$$S_i = \{1, T_i, T_i T_{i-1}, \dots, T_i T_{i-1} \dots T_1\},$$

...

$$S_n = \{1, T_n, T_n T_{n-1}, \dots, T_n T_{n-1} \dots T_1\}.$$

Note the property that $V_i \in S_i$ implies $T_{i+1} V_i \in S_{i+1}$.

Consider the set of monomials $M = U_1 \cdot U_2 \cdot \dots \cdot U_n$ for all possible choices of $U_i \in S_i, i = 1, \dots, n$. We shall say that these monomials are in *normal form*. There are $(n+1)!$ of them.

We claim that these monomials M generate H_{n+1} as a K -space. Consequently, $\dim_K H_{n+1} \leq (n+1)!$ and also $\dim_K \{H_n \oplus H_n \otimes H_n\} \leq (n+1)!$, where the tensor product is over H_{n-1} as above.

Proof. We may assume by induction that the claim holds for H_n . As H_{n+1} is generated over K by monomials M_0 and $M = M_1 T_n M_2$, where M_0, M_1, M_2 are monomials in T_1, \dots, T_{n-1} , and as the induction hypothesis makes the case of M_0 clear, we concentrate on $M = M_1 T_n M_2$. By induction, M_2 is a K -linear combination of monomials of the form $V_1 \cdot V_2 \cdot \dots \cdot V_{n-1}$, with $V_i \in S_i$ for $i = 1, \dots, n-1$. We have

$$M_1 T_n V_1 V_2 \dots V_{n-1} = M'_1 T_n V_{n-1} = M'_1 U_n,$$

with $U_n = T_n V_{n-1} \in S_n$. By induction again, M'_1 is a K -linear combination of monomials of the form $U_1 \cdot U_2 \cdot \dots \cdot U_{n-1}$ with $U_i \in S_i$. Thus M is a K -linear combination of monomials $U_1 \cdot U_2 \cdot \dots \cdot U_n$ as desired and $\dim_K H_{n+1} \leq (n+1)!$.

This shows also that $H_n \otimes_{H_{n-1}} H_n$ is spanned over K by the subspaces $H_n \otimes U_{n-1}$ with $U_{n-1} \in S_{n-1}$. Therefore, its K -dimension is at most $n! \cdot n$, so that the proof of claim 3 is complete.

Remark. Let \mathfrak{S}_{n+1} be the symmetric group on $\{1, \dots, n+1\}$, and denote by s_i the transposition $(i, i+1)$. The same argument as above shows that any $\pi \in \mathfrak{S}_{n+1}$ can be written as a product $w_1 \cdot w_2 \cdot \dots \cdot w_n$, where

$$w_i \in \{1, s_i, s_i s_{i-1}, \dots, s_i s_{i-1} \dots s_1\}.$$

We shall use this remark presently in the proof of the following claim 4.

Exercise. Deduce from the remark that \mathfrak{S}_{n+1} has a presentation on generators s_1, \dots, s_n with the relations

$$\begin{aligned} s_i s_j &= s_j s_i & \text{whenever } |i-j| \geq 2 & \text{ with } i, j = 1, \dots, n, \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} & \text{for } i = 1, \dots, n-1, \\ s_i^2 &= 1 & \text{for } i = 1, \dots, n. \end{aligned}$$

CLAIM 4. The monomials in normal form $M = U_1 \cdot U_2 \cdot \dots \cdot U_n$, with $U_i \in S_i$ for $i = 1, \dots, n$ are K -linearly independent. Also, the map φ is an isomorphism.

Proof. Denote by $l: \mathfrak{S}_{n+1} \rightarrow N$ the word length in \mathfrak{S}_{n+1} , relative to the generators $\{s_1, s_2, \dots, s_n\}$. For $i \in \{1, \dots, n\}$, define $L_i \in \text{End}_K(K\mathfrak{S}_{n+1})$ by

$$L_i(\pi) = \begin{cases} s_i\pi & \text{if } l(s_i\pi) > l(\pi), \\ qs_i\pi + (q-1)\pi & \text{if } l(s_i\pi) < l(\pi), \end{cases}$$

for every $\pi \in \mathfrak{S}_{n+1}$.

The crucial fact is the following

ASSERTION. *There is an algebra map $L: H_{n+1} \rightarrow \text{End}_K(K\mathfrak{S}_{n+1})$ such that $L(T_i) = L_i$ for $i = 1, \dots, n$.*

To prove the assertion, we have to check that the endomorphisms $L_i \in \text{End}_K(K\mathfrak{S}_{n+1})$ satisfy the defining relations of the Hecke algebra H_{n+1} . For this, see the following three claims.

Assuming the assertion, consider a monomial in normal form $M = U_1 \cdot U_2 \dots U_n$ as above. Then, $L(M)$ maps $1 \in K\mathfrak{S}_{n+1}$ to $w_1 \cdot w_2 \dots w_n$, where $w_i = s_i s_{i-1} \dots s_{i-j}$ if $U_i = T_i T_{i-1} \dots T_{i-j}$. The remark after claim 3 now shows that any of the $(n+1)!$ elements of \mathfrak{S}_{n+1} is of the form $w_1 \cdot w_2 \dots w_n$, so that these elements are K -linearly independent in $K\mathfrak{S}_{n+1}$. But, as the map from H_{n+1} to $K\mathfrak{S}_{n+1}$ which sends x to $L(x)(1)$ is K -linear, this implies that the elements $M = U_1 \cdot U_2 \dots U_n$ in normal form must also be linearly independent. Hence, $\dim_K H_{n+1} = (n+1)!$.

Now, a dimension count shows that the surjective map φ is an isomorphism.

It remains to prove the above assertion: The L_i 's satisfy the defining relations for H_{n+1} .

CLAIM 5. $L_i^2 = (q-1)L_i + q$ for $i = 1, \dots, n$.

Proof. Let $\pi \in \mathfrak{S}_{n+1}$. If $l(s_i\pi) > l(\pi)$, then

$$\begin{aligned} L_i^2(\pi) &= L_i(s_i\pi) = qs_i^2\pi + (q-1)s_i\pi \\ &= (q-1)s_i\pi + q\pi = ((q-1)L_i + q)(\pi). \end{aligned}$$

If on the other hand, $l(s_i\pi) < l(\pi)$, set $\pi' = s_i\pi$ and observe that $l(s_i\pi') > l(\pi')$. Thus,

$$\begin{aligned} L_i^2(\pi) &= L_i(qs_i\pi + (q-1)\pi) = L_i(q\pi' + (q-1)\pi) \\ &= qs_i\pi' + (q-1)L_i(\pi) = ((q-1)L_i + q)(\pi). \end{aligned}$$

The next claim will be used in proving the last two types of relations for the endomorphisms L_i .

CLAIM 6. For $j = 1, \dots, n$ define $R_j \in \text{End}_K(K\mathfrak{S}_{n+1})$ by

$$R_j(\pi) = \begin{cases} \pi s_j & \text{if } l(\pi s_j) > l(\pi), \\ q\pi s_j + (q-1)\pi & \text{if } l(\pi s_j) < l(\pi). \end{cases}$$

Then, $L_i R_j = R_j L_i$ for all $i, j \in \{1, \dots, n\}$.

Proof. Choose $i, j \in \{1, \dots, n\}$ and $\pi \in \mathfrak{S}_{n+1}$. The proof that $L_i R_j(\pi) = R_j L_i(\pi)$ is by direct verification from the definitions of the operators L_i, R_j and is divided into six cases.

$$(6.1) \quad l(s_i \pi s_j) = l(\pi) + 2,$$

$$(6.2) \quad l(s_i \pi s_j) = l(\pi) - 2,$$

$$(6.3)-(6.6) \quad l(s_i \pi s_j) = l(\pi) \quad \text{and}$$

$$l(s_i \pi) = l(\pi) + \varepsilon, \quad \text{where } \varepsilon = \pm 1,$$

$$l(\pi s_j) = l(\pi) + \varepsilon', \quad \text{where } \varepsilon' = \pm 1.$$

The first two cases are straightforward calculations.

Among the last four cases, two are also trivial, namely those with $\varepsilon \neq \varepsilon'$. There remain the two cases with $\varepsilon = \varepsilon' = \pm 1$. Then, the *exchange lemma* applied to the symmetric group viewed as a Coxeter group (on the generators s_1, \dots, s_n) implies that in these cases we have $s_i \pi = \pi s_j$. (If $\varepsilon = \varepsilon' = +1$, this equality is given as property C in Bourbaki, Groupes et Algèbres de Lie, Chap. IV, n° 1.7. If $\varepsilon = \varepsilon' = -1$, the same property yields $s_i(\pi s_j) = (\pi s_j)s_j$.) This is just what is needed to complete the verification of $L_i R_j(\pi) = R_j L_i(\pi)$.

CLAIM 7. $L_i L_j = L_j L_i$ whenever $|i-j| \geq 2$,

$$L_i L_{i+1} L_i = L_{i+1} L_i L_{i+1}.$$

Proof. Let $\pi \in \mathfrak{S}_{n+1}$. Write $\pi = s_{i_1} \cdot s_{i_2} \dots s_{i_r}$ in reduced form, i.e. with $r = l(\pi)$. We thus have $\pi = R_{i_r} R_{i_{r-1}} \dots R_{i_1}(1)$.

Setting $R = R_{i_r} \dots R_{i_1}$, we have

$$L_i L_j(\pi) = L_i L_j R(1) = R L_i L_j(1) \quad \text{by claim 6,}$$

$$= R(s_i s_j) = R(s_j s_i) \quad \text{since } |i-j| \geq 2, \quad \text{and thus}$$

$$L_i L_j(\pi) = L_j L_i(\pi).$$

Since this holds for every $\pi \in \mathfrak{S}_{n+1}$, one has $L_i L_j = L_j L_i$.

A similar calculation, based on the same principle, proves that $L_i L_{i+1} L_i = L_{i+1} L_i L_{i+1}$ for $i = 1, \dots, n-1$.

This completes the proof of Proposition 4.1.