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§ 4. HECKE ALGEBRAS

In this section we isolate the classical facts about Hecke algebras which we will need in the next two sections in order to prove the existence of P . The knowledgeable reader can thus skip this paragraph and proceed directly to § 5.

Let K be a field and let $q \in K$ be some element of K .

The Hecke algebra H_n over K corresponding to q is the associative K -algebra with unit 1, generated by T_1, \dots, T_{n-1} subject to the following relations

$$T_i T_j = T_j T_i \quad \text{whenever} \quad |i-j| \geq 2,$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \quad \text{and}$$

$$T_i^2 = (q-1)T_i + q$$

for all $i, j \in \{1, \dots, n-1\}$, with of course $i \leq n-2$ for the second family of relations.

We see that there is a natural map $H_n \rightarrow H_{n+1}$ of K -algebras which make H_{n+1} a (H_n, H_n) -bimodule. We think of $q \in K$ as being fixed once and for all.

Consider also the (H_n, H_n) -bimodule $H_n \oplus H_n \otimes_{H_{n-1}} H_n$.

PROPOSITION 4.1. *There is a natural map of (H_n, H_n) -bimodules*

$$\varphi: H_n \oplus H_n \otimes_{H_{n-1}} H_n \rightarrow H_{n+1}$$

given by $\varphi(a + \sum_i b_i \otimes c_i) = a + \sum_i b_i T_n c_i$.

Moreover, φ is an isomorphism.

The proof of this proposition will occupy the remainder of this section. We have divided it into seven claims.

CLAIM 1. *The map φ is well defined.*

Proof. If $u \in H_{n-1}$, then

$$\varphi(bu \otimes c) = bu T_n c, \quad \text{and} \quad \varphi(b \otimes uc) = b T_n uc.$$

But u is a K -linear combination of monomials in T_1, \dots, T_{n-2} which commute with T_n in H_{n+1} . Hence, $bu T_n c = b T_n uc$, and so φ is well defined.

CLAIM 2. *The map φ is surjective.*

We have to show that H_{n+1} is generated as a vector space over K by the monomials with at most one occurrence of T_n .

The proof will be by induction on n . Let M be a monomial in T_1, \dots, T_n with two occurrences of T_n at least. Displaying two consecutive occurrences of T_n in M , we write $M = M_1 T_n M_2 T_n M_3$, where we can assume that M_2 is a monomial in T_1, \dots, T_{n-1} only. Assume by induction that M_2 contains T_{n-1} at most once. If M_2 does not contain T_{n-1} at all, then

$$M = M_1 M_2 T_n^2 M_3 = (q-1)M_1 M_2 T_n M_3 + qM_1 M_2 M_3,$$

reducing the number of occurrences of T_n in each new monomial. If M_2 contains T_{n-1} exactly once, we can write $M_2 = M' T_{n-1} M''$, with M', M'' monomials in T_1, \dots, T_{n-2} and then,

$$M = M_1 M' T_n T_{n-1} T_n M'' M_3,$$

using the fact that T_1, \dots, T_{n-2} commute with T_n . But now, $T_n T_{n-1} T_n = T_{n-1} T_n T_{n-1}$ yields

$$M = M_1 M' T_{n-1} T_n T_{n-1} M'' M_3,$$

reducing again the number of occurrences of T_n .

Hence, every element of H_{n+1} is a sum $a + \sum_i b_i T_n c_i$ with a, b_i, c_i coming from H_n and it is now clear that φ is surjective.

CLAIM 3. *Monomials in normal form generate H_{n+1} over K .*

We have actually proved a little more than was stated in Claim 2. Consider the following lists of monomials:

$$S_1 = \{1, T_1\},$$

$$S_2 = \{1, T_2, T_2 T_1\},$$

$$S_3 = \{1, T_3, T_3 T_2, T_3 T_2 T_1\},$$

...

$$S_i = \{1, T_i, T_i T_{i-1}, \dots, T_i T_{i-1} \dots T_1\},$$

...

$$S_n = \{1, T_n, T_n T_{n-1}, \dots, T_n T_{n-1} \dots T_1\}.$$

Note the property that $V_i \in S_i$ implies $T_{i+1} V_i \in S_{i+1}$.

Consider the set of monomials $M = U_1 \cdot U_2 \cdot \dots \cdot U_n$ for all possible choices of $U_i \in S_i, i = 1, \dots, n$. We shall say that these monomials are in *normal form*. There are $(n+1)!$ of them.

We claim that these monomials M generate H_{n+1} as a K -space. Consequently, $\dim_K H_{n+1} \leq (n+1)!$ and also $\dim_K \{H_n \oplus H_n \otimes H_n\} \leq (n+1)!$, where the tensor product is over H_{n-1} as above.

Proof. We may assume by induction that the claim holds for H_n . As H_{n+1} is generated over K by monomials M_0 and $M = M_1 T_n M_2$, where M_0, M_1, M_2 are monomials in T_1, \dots, T_{n-1} , and as the induction hypothesis makes the case of M_0 clear, we concentrate on $M = M_1 T_n M_2$. By induction, M_2 is a K -linear combination of monomials of the form $V_1 \cdot V_2 \cdot \dots \cdot V_{n-1}$, with $V_i \in S_i$ for $i = 1, \dots, n-1$. We have

$$M_1 T_n V_1 V_2 \dots V_{n-1} = M'_1 T_n V_{n-1} = M'_1 U_n,$$

with $U_n = T_n V_{n-1} \in S_n$. By induction again, M'_1 is a K -linear combination of monomials of the form $U_1 \cdot U_2 \cdot \dots \cdot U_{n-1}$ with $U_i \in S_i$. Thus M is a K -linear combination of monomials $U_1 \cdot U_2 \cdot \dots \cdot U_n$ as desired and $\dim_K H_{n+1} \leq (n+1)!$.

This shows also that $H_n \otimes_{H_{n-1}} H_n$ is spanned over K by the subspaces $H_n \otimes U_{n-1}$ with $U_{n-1} \in S_{n-1}$. Therefore, its K -dimension is at most $n! \cdot n$, so that the proof of claim 3 is complete.

Remark. Let \mathfrak{S}_{n+1} be the symmetric group on $\{1, \dots, n+1\}$, and denote by s_i the transposition $(i, i+1)$. The same argument as above shows that any $\pi \in \mathfrak{S}_{n+1}$ can be written as a product $w_1 \cdot w_2 \cdot \dots \cdot w_n$, where

$$w_i \in \{1, s_i, s_i s_{i-1}, \dots, s_i s_{i-1} \dots s_1\}.$$

We shall use this remark presently in the proof of the following claim 4.

Exercise. Deduce from the remark that \mathfrak{S}_{n+1} has a presentation on generators s_1, \dots, s_n with the relations

$$\begin{aligned} s_i s_j &= s_j s_i & \text{whenever } |i - j| \geq 2 & \text{ with } i, j = 1, \dots, n, \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} & \text{for } i = 1, \dots, n-1, \\ s_i^2 &= 1 & \text{for } i = 1, \dots, n. \end{aligned}$$

CLAIM 4. *The monomials in normal form $M = U_1 \cdot U_2 \cdot \dots \cdot U_n$, with $U_i \in S_i$ for $i = 1, \dots, n$ are K -linearly independent. Also, the map φ is an isomorphism.*

Proof. Denote by $l: \mathfrak{S}_{n+1} \rightarrow N$ the word length in \mathfrak{S}_{n+1} , relative to the generators $\{s_1, s_2, \dots, s_n\}$. For $i \in \{1, \dots, n\}$, define $L_i \in \text{End}_K(K\mathfrak{S}_{n+1})$ by

$$L_i(\pi) = \begin{cases} s_i\pi & \text{if } l(s_i\pi) > l(\pi), \\ qs_i\pi + (q-1)\pi & \text{if } l(s_i\pi) < l(\pi), \end{cases}$$

for every $\pi \in \mathfrak{S}_{n+1}$.

The crucial fact is the following

ASSERTION. *There is an algebra map $L: H_{n+1} \rightarrow \text{End}_K(K\mathfrak{S}_{n+1})$ such that $L(T_i) = L_i$ for $i = 1, \dots, n$.*

To prove the assertion, we have to check that the endomorphisms $L_i \in \text{End}_K(K\mathfrak{S}_{n+1})$ satisfy the defining relations of the Hecke algebra H_{n+1} . For this, see the following three claims.

Assuming the assertion, consider a monomial in normal form $M = U_1 \cdot U_2 \dots U_n$ as above. Then, $L(M)$ maps $1 \in K\mathfrak{S}_{n+1}$ to $w_1 \cdot w_2 \dots w_n$, where $w_i = s_i s_{i-1} \dots s_{i-j}$ if $U_i = T_i T_{i-1} \dots T_{i-j}$. The remark after claim 3 now shows that any of the $(n+1)!$ elements of \mathfrak{S}_{n+1} is of the form $w_1 \cdot w_2 \dots w_n$, so that these elements are K -linearly independent in $K\mathfrak{S}_{n+1}$. But, as the map from H_{n+1} to $K\mathfrak{S}_{n+1}$ which sends x to $L(x)(1)$ is K -linear, this implies that the elements $M = U_1 \cdot U_2 \dots U_n$ in normal form must also be linearly independent. Hence, $\dim_K H_{n+1} = (n+1)!$.

Now, a dimension count shows that the surjective map φ is an isomorphism.

It remains to prove the above assertion: The L_i 's satisfy the defining relations for H_{n+1} .

CLAIM 5. $L_i^2 = (q-1)L_i + q$ for $i = 1, \dots, n$.

Proof. Let $\pi \in \mathfrak{S}_{n+1}$. If $l(s_i\pi) > l(\pi)$, then

$$\begin{aligned} L_i^2(\pi) &= L_i(s_i\pi) = qs_i^2\pi + (q-1)s_i\pi \\ &= (q-1)s_i\pi + q\pi = ((q-1)L_i + q)(\pi). \end{aligned}$$

If on the other hand, $l(s_i\pi) < l(\pi)$, set $\pi' = s_i\pi$ and observe that $l(s_i\pi') > l(\pi')$. Thus,

$$\begin{aligned} L_i^2(\pi) &= L_i(qs_i\pi + (q-1)\pi) = L_i(q\pi' + (q-1)\pi) \\ &= qs_i\pi' + (q-1)L_i(\pi) = ((q-1)L_i + q)(\pi). \end{aligned}$$

The next claim will be used in proving the last two types of relations for the endomorphisms L_i .

CLAIM 6. For $j = 1, \dots, n$ define $R_j \in \text{End}_K(K\mathfrak{S}_{n+1})$ by

$$R_j(\pi) = \begin{cases} \pi s_j & \text{if } l(\pi s_j) > l(\pi), \\ q\pi s_j + (q-1)\pi & \text{if } l(\pi s_j) < l(\pi). \end{cases}$$

Then, $L_i R_j = R_j L_i$ for all $i, j \in \{1, \dots, n\}$.

Proof. Choose $i, j \in \{1, \dots, n\}$ and $\pi \in \mathfrak{S}_{n+1}$. The proof that $L_i R_j(\pi) = R_j L_i(\pi)$ is by direct verification from the definitions of the operators L_i, R_j and is divided into six cases.

$$(6.1) \quad l(s_i \pi s_j) = l(\pi) + 2,$$

$$(6.2) \quad l(s_i \pi s_j) = l(\pi) - 2,$$

$$(6.3)-(6.6) \quad l(s_i \pi s_j) = l(\pi) \quad \text{and}$$

$$l(s_i \pi) = l(\pi) + \varepsilon, \quad \text{where } \varepsilon = \pm 1,$$

$$l(\pi s_j) = l(\pi) + \varepsilon', \quad \text{where } \varepsilon' = \pm 1.$$

The first two cases are straightforward calculations.

Among the last four cases, two are also trivial, namely those with $\varepsilon \neq \varepsilon'$. There remain the two cases with $\varepsilon = \varepsilon' = \pm 1$. Then, the *exchange lemma* applied to the symmetric group viewed as a Coxeter group (on the generators s_1, \dots, s_n) implies that in these cases we have $s_i \pi = \pi s_j$. (If $\varepsilon = \varepsilon' = +1$, this equality is given as property C in Bourbaki, Groupes et Algèbres de Lie, Chap. IV, n° 1.7. If $\varepsilon = \varepsilon' = -1$, the same property yields $s_i(\pi s_j) = (\pi s_j)s_j$.) This is just what is needed to complete the verification of $L_i R_j(\pi) = R_j L_i(\pi)$.

CLAIM 7. $L_i L_j = L_j L_i$ whenever $|i-j| \geq 2$,

$$L_i L_{i+1} L_i = L_{i+1} L_i L_{i+1}.$$

Proof. Let $\pi \in \mathfrak{S}_{n+1}$. Write $\pi = s_{i_1} \cdot s_{i_2} \dots s_{i_r}$ in reduced form, i.e. with $r = l(\pi)$. We thus have $\pi = R_{i_r} R_{i_{r-1}} \dots R_{i_1}(1)$.

Setting $R = R_{i_r} \dots R_{i_1}$, we have

$$L_i L_j(\pi) = L_i L_j R(1) = R L_i L_j(1) \quad \text{by claim 6,}$$

$$= R(s_i s_j) = R(s_j s_i) \quad \text{since } |i-j| \geq 2, \quad \text{and thus}$$

$$L_i L_j(\pi) = L_j L_i(\pi).$$

Since this holds for every $\pi \in \mathfrak{S}_{n+1}$, one has $L_i L_j = L_j L_i$.

A similar calculation, based on the same principle, proves that $L_i L_{i+1} L_i = L_{i+1} L_i L_{i+1}$ for $i = 1, \dots, n-1$.

This completes the proof of Proposition 4.1.