Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 32 (1986)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: ON THE JONES POLYNOMIAL Swiss Seminar in Berne

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Kapitel: §4. Hecke algebras

DOI: https://doi.org/10.5169/seals-55091

§ 4. HECKE ALGEBRAS

In this section we isolate the classical facts about Hecke algebras which we will need in the next two sections in order to prove the existence of P. The knowledgeable reader can thus skip this paragraph and proceed directly to § 5.

Let K be a field and let $q \in K$ be some element of K.

The Hecke algebra H_n over K corresponding to q is the associative K-algebra with unit 1, generated by $T_1, \ldots, T_{n-1}$ subject to the following relations

$$\begin{aligned} T_i T_j &= T_j T_i \quad \text{whenever} \quad |i-j| \geqslant 2, \\ T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1}, \quad \text{and} \\ T_i^2 &= (q-1)T_i + q \end{aligned}$$

for all $i, j \in \{1, \ldots, n-1\}$, with of course $i \leqslant n-2$ for the second family of relations.

We see that there is a natural map $H_n \to H_{n+1}$ of K-algebras which make H_{n+1} a (H_n, H_n)-bimodule. We think of $q \in K$ as being fixed once and for all.

Consider also the (H_n, H_n)-bimodule $H_n \oplus H_n \otimes_{H_{n-1}} H_n$.

PROPOSITION 4.1. *There is a natural map of* (H_n, H_n)*-bimodules*

$$\varphi : H_n \oplus H_n \otimes_{H_{n-1}} H_n \to H_{n+1}$$

given by $\varphi(a + \Sigma_i b_i \otimes c_i) = a + \Sigma_i b_i T_n c_i$.

Moreover, φ *is an isomorphism.*

The proof of this proposition will occupy the remainder of this section. We have divided it into seven claims.

CLAIM 1. *The map* φ *is well defined.*

Proof. If $u \in H_{n-1}$, then

$$\varphi(bu \otimes c) = buT_n c, \quad \text{and} \quad \varphi(b \otimes uc) = bT_n uc.$$

But u is a K-linear combination of monomials in $T_1, \ldots, T_{n-2}$ which commute with T_n in H_{n+1}. Hence, $buT_n c = bT_n uc$, and so φ is well defined.

CLAIM 2. *The map* φ *is surjective.*

We have to show that H_{n+1} is generated as a vector space over K by the monomials with at most one occurence of T_n.

The proof will be by induction on n. Let M be a monomial in $T_1, \ldots, T_n$ with two occurences of T_n at least. Displaying two consecutive occurences of T_n in M, we write $M = M_1 T_n M_2 T_n M_3$, where we can assume that M_2 is a monomial in $T_1, \ldots, T_{n-1}$ only. Assume *by induction* that M_2 contains T_{n-1} at most once. If M_2 does not contain T_{n-1} at all, then

$$M = M_1 M_2 T_n^2 M_3 = (q-1) M_1 M_2 T_n M_3 + q M_1 M_2 M_3 ,$$

reducing the number of occurences of T_n in each new monomial. If M_2 contains T_{n-1} exactly once, we can write $M_2 = M' T_{n-1} M''$, with M', M'' monomials in $T_1, \ldots, T_{n-2}$ and then,

$$M = M_1 M' T_n T_{n-1} T_n M'' M_3 ,$$

using the fact that $T_1, \ldots, T_{n-2}$ commute with T_n. But now, $T_n T_{n-1} T_n = T_{n-1} T_n T_{n-1}$ yields

$$M = M_1 M' T_{n-1} T_n T_{n-1} M'' M_3 ,$$

reducing again the number of occurences of T_n.

Hence, every element of H_{n+1} is a sum $a + \Sigma_i b_i T_n c_i$ with a, b_i, c_i coming from H_n and it is now clear that φ is surjective.

CLAIM 3. *Monomials in normal form generate* H_{n+1} *over* K.

We have actually proved a little more than was stated in Claim 2. Consider the following lists of monomials:

$$\begin{aligned}
S_1 &= \{1, T_1\} , \\
S_2 &= \{1, T_2, T_2 T_1\} , \\
S_3 &= \{1, T_3, T_3 T_2, T_3 T_2 T_1\} , \\
&\ldots \\
S_i &= \{1, T_i, T_i T_{i-1}, \ldots, T_i T_{i-1} \ldots T_1\} , \\
&\ldots \\
S_n &= \{1, T_n, T_n T_{n-1}, \ldots, T_n T_{n-1} \ldots T_1\} .
\end{aligned}$$

Note the property that $V_i \in S_i$ implies $T_{i+1} V_i \in S_{i+1}$.

Consider the set of monomials $M = U_1 . U_2 . \dots . U_n$ for all possible choices of $U_i \in S_i$, $i = 1, \dots, n$. We shall say that these monomials are in *normal form*. There are $(n+1)!$ of them.

We claim that these monomials M generate H_{n+1} as a K-space. Consequently, $\dim_K H_{n+1} \leqslant (n+1)!$ and also $\dim_K\{H_n \oplus H_n \otimes H_n\} \leqslant (n+1)!$, where the tensor product is over H_{n-1} as above.

Proof. We may assume *by induction* that the claim holds for H_n. As H_{n+1} is generated over K by monomials M_0 and $M = M_1 T_n M_2$, where M_0, M_1, M_2 are monomials in $T_1, \dots, T_{n-1}$, and as the induction hypothesis makes the case of M_0 clear, we concentrate on $M = M_1 T_n M_2$. By induction, M_2 is a K-linear combination of monomials of the form $V_1 . V_2 . \dots . V_{n-1}$, with $V_i \in S_i$ for $i = 1, \dots, n-1$. We have

$$M_1 T_n V_1 V_2 \dots V_{n-1} = M'_1 T_n V_{n-1} = M'_1 U_n ,$$

with $U_n = T_n V_{n-1} \in S_n$. By induction again, M'_1 is a K-linear combination of monomials of the form $U_1 . U_2 . \dots U_{n-1}$ with $U_i \in S_i$. Thus M is a K-linear combination of monomials $U_1 . U_2 . \dots U_n$ as desired and $\dim_K H_{n+1} \leqslant (n+1)!$.

This shows also that $H_n \otimes_{H_{n-1}} H_n$ is spanned over K by the subspaces $H_n \otimes U_{n-1}$ with $U_{n-1} \in S_{n-1}$. Therefore, its K-dimension is at most $n! . n$, so that the proof of claim 3 is complete.

Remark. Let $\mathfrak{S}_{n+1}$ be the symmetric group on $\{1, \dots, n+1\}$, and denote by s_i the transposition $(i, i+1)$. The same argument as above shows that any $\pi \in \mathfrak{S}_{n+1}$ can be written as a product $w_1 . w_2 \dots . w_n$, where

$$w_i \in \{1, s_i, s_i s_{i-1}, \dots, s_i s_{i-1} \dots s_1\} .$$

We shall use this remark presently in the proof of the following claim 4.

Exercise. Deduce from the remark that $\mathfrak{S}_{n+1}$ has a presentation on generators $s_1, \dots, s_n$ with the relations

$$\begin{array}{llll} s_i s_j = s_j s_i & \text{whenever} \quad |i-j| \geqslant 2 & \text{with} & i,j = 1, \dots, n , \\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} & \text{for} \quad i = 1, \dots, n-1 , & & \\ s_i^2 = 1 & \text{for} \quad i = 1, \dots, n . & & \end{array}$$

CLAIM 4. *The monomials in normal form* $M = U_1 . U_2 \dots . U_n$, *with* $U_i \in S_i$ *for* $i = 1, \dots, n$ *are* K-*linearly independent. Also, the map* φ *is an isomorphism.*

Proof. Denote by $l: \mathfrak{S}_{n+1} \to N$ the word length in $\mathfrak{S}_{n+1}$, relative to the generators $\{s_1, s_2, ..., s_n\}$. For $i \in \{1, ..., n\}$, define $L_i \in \mathrm{End}_K(K\mathfrak{S}_{n+1})$ by

$$L_i(\pi) = \begin{cases} s_i\pi & \text{if} \quad l(s_i\pi) > l(\pi)\,, \\ qs_i\pi + (q-1)\pi & \text{if} \quad l(s_i\pi) < l(\pi)\,, \end{cases}$$

for every $\pi \in \mathfrak{S}_{n+1}$.

The crucial fact is the following

ASSERTION. *There is an algebra map* $L: H_{n+1} \to \mathrm{End}_K(K\mathfrak{S}_{n+1})$ *such that* $L(T_i) = L_i$ *for* $i = 1, ..., n$.

To prove the assertion, we have to check that the endomorphisms $L_i \in \mathrm{End}_K(K\mathfrak{S}_{n+1})$ satisfy the defining relations of the Hecke algebra H_{n+1}. For this, see the following three claims.

Assuming the assertion, consider a monomial in normal form $M = U_1 . U_2 .... U_n$ as above. Then, $L(M)$ maps $1 \in K\mathfrak{S}_{n+1}$ to $w_1 . w_2 .... w_n$, where $w_i = s_i s_{i-1} ... s_{i-j}$ if $U_i = T_i T_{i-1} ... T_{i-j}$. The remark after claim 3 now shows that any of the $(n+1)!$ elements of $\mathfrak{S}_{n+1}$ is of the form $w_1 . w_2 .... w_n$, so that these elements are K-linearly independent in $K\mathfrak{S}_{n+1}$. But, as the map from H_{n+1} to $K\mathfrak{S}_{n+1}$ which sends x to $L(x)(1)$ is K-linear, this implies that the elements $M = U_1 . U_2 .... U_n$ in normal form must also be linearly independent. Hence, $\dim_K H_{n+1} = (n+1)!$.

Now, a dimension count shows that the surjective map φ is an isomorphism.

It remains to prove the above assertion: The L_i's satisfy the defining relations for H_{n+1}.

CLAIM 5. $L_i^2 = (q-1)L_i + q$ *for* $i = 1, ..., n$.

Proof. Let $\pi \in \mathfrak{S}_{n+1}$. If $l(s_i\pi) > l(\pi)$, then

$$\begin{aligned} L_i^2(\pi) &= L_i(s_i\pi) = qs_i^2\pi + (q-1)s_i\pi \\ &= (q-1)s_i\pi + q\pi = \big((q-1)L_i + q\big)(\pi)\,. \end{aligned}$$

If on the other hand, $l(s_i\pi) < l(\pi)$, set $\pi' = s_i\pi$ and observe that $l(s_i\pi') > l(\pi')$. Thus,

$$\begin{aligned} L_i^2(\pi) &= L_i\big(qs_i\pi + (q-1)\pi\big) = L_i\big(q\pi' + (q-1)\pi\big) \\ &= qs_i\pi' + (q-1)L_i(\pi) = \big((q-1)L_i + q\big)(\pi)\,. \end{aligned}$$

The next claim will be used in proving the last two types of relations for the endomorphisms L_i.

CLAIM 6. *For* $j = 1, \ldots, n$ *define* $R_j \in \operatorname{End}_K(K\mathfrak{S}_{n+1})$ *by*

$$R_j(\pi) = \begin{cases} \pi s_j & \text{if} \quad l(\pi s_j) > l(\pi), \\ q\pi s_j + (q-1)\pi & \text{if} \quad l(\pi s_j) < l(\pi). \end{cases}$$

Then, $L_i R_j = R_j L_i$ *for all* $i, j \in \{1, \ldots, n\}$.

Proof. Choose $i, j \in \{1, \ldots, n\}$ and $\pi \in \mathfrak{S}_{n+1}$. The proof that $L_i R_j(\pi) = R_j L_i(\pi)$ is by direct verification from the definitions of the operators L_i, R_j and is divided into six cases.

(6.1) $\quad l(s_i \pi s_j) = l(\pi) + 2,$

(6.2) $\quad l(s_i \pi s_j) = l(\pi) - 2,$

(6.3)-(6.6) $\quad l(s_i \pi s_j) = l(\pi)$ and

$l(s_i \pi) = l(\pi) + \varepsilon,$ where $\varepsilon = \pm 1,$

$l(\pi s_j) = l(\pi) + \varepsilon',$ where $\varepsilon' = \pm 1.$

The first two cases are straightforward calculations.

Among the last four cases, two are also trivial, namely those with $\varepsilon \neq \varepsilon'$. There remain the two cases with $\varepsilon = \varepsilon' = \pm 1$. Then, the *exchange lemma* applied to the symmetric group viewed as a Coxeter group (on the generators $s_1, \ldots, s_n$) implies that in these cases we have $s_i \pi = \pi s_j$. (If $\varepsilon = \varepsilon' = +1$, this equality is given as property C in Bourbaki, Groupes et Algèbres de Lie, Chap. IV, n° 1.7. If $\varepsilon = \varepsilon' = -1$, the same property yields $s_i(\pi s_j) = (\pi s_j) s_j$.) This is just what is needed to complete the verification of $L_i R_j(\pi) = R_j L_i(\pi)$.

CLAIM 7. $L_i L_j = L_j L_i$ *whenever* $|i - j| \geqslant 2$,

$$L_i L_{i+1} L_i = L_{i+1} L_i L_{i+1}.$$

Proof. Let $\pi \in \mathfrak{S}_{n+1}$. Write $\pi = s_{i_1} . s_{i_2} \ldots . s_{i_r}$ in reduced form, i.e. with $r = l(\pi)$. We thus have $\pi = R_{i_r} R_{i_{r-1}} \ldots R_{i_1}(1)$.

Setting $R = R_{i_r} \ldots R_{i_1}$, we have

$$\begin{aligned} L_i L_j(\pi) &= L_i L_j R(1) = R L_i L_j(1) \quad \text{by claim 6}, \\ &= R(s_i s_j) = R(s_j s_i) \quad \text{since} \quad |i-j| \geqslant 2, \quad \text{and thus} \\ L_i L_j(\pi) &= L_j L_i(\pi). \end{aligned}$$

Since this holds for every $\pi \in \mathfrak{S}_{n+1}$, one has $L_i L_j = L_j L_i$.

A similar calculation, based on the same principle, proves that $L_i L_{i+1} L_i = L_{i+1} L_i L_{i+1}$ for $i = 1, \ldots, n-1$.

This completes the proof of Proposition 4.1.