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## § 4. HECKE ALGEBRAS

In this section we isolate the classical facts about Hecke algebras which we will need in the next two sections in order to prove the existence of  $P$ . The knowledgeable reader can thus skip this paragraph and proceed directly to § 5.

Let  $K$  be a field and let  $q \in K$  be some element of  $K$ .

The Hecke algebra  $H_n$  over  $K$  corresponding to  $q$  is the associative  $K$ -algebra with unit 1, generated by  $T_1, \dots, T_{n-1}$  subject to the following relations

$$T_i T_j = T_j T_i \quad \text{whenever} \quad |i-j| \geq 2,$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \quad \text{and}$$

$$T_i^2 = (q-1)T_i + q$$

for all  $i, j \in \{1, \dots, n-1\}$ , with of course  $i \leq n-2$  for the second family of relations.

We see that there is a natural map  $H_n \rightarrow H_{n+1}$  of  $K$ -algebras which make  $H_{n+1}$  a  $(H_n, H_n)$ -bimodule. We think of  $q \in K$  as being fixed once and for all.

Consider also the  $(H_n, H_n)$ -bimodule  $H_n \oplus H_n \otimes_{H_{n-1}} H_n$ .

**PROPOSITION 4.1.** *There is a natural map of  $(H_n, H_n)$ -bimodules*

$$\varphi: H_n \oplus H_n \otimes_{H_{n-1}} H_n \rightarrow H_{n+1}$$

given by  $\varphi(a + \sum_i b_i \otimes c_i) = a + \sum_i b_i T_n c_i$ .

Moreover,  $\varphi$  is an isomorphism.

The proof of this proposition will occupy the remainder of this section. We have divided it into seven claims.

**CLAIM 1.** *The map  $\varphi$  is well defined.*

*Proof.* If  $u \in H_{n-1}$ , then

$$\varphi(bu \otimes c) = bu T_n c, \quad \text{and} \quad \varphi(b \otimes uc) = b T_n uc.$$

But  $u$  is a  $K$ -linear combination of monomials in  $T_1, \dots, T_{n-2}$  which commute with  $T_n$  in  $H_{n+1}$ . Hence,  $bu T_n c = b T_n uc$ , and so  $\varphi$  is well defined.

**CLAIM 2.** *The map  $\varphi$  is surjective.*

We have to show that  $H_{n+1}$  is generated as a vector space over  $K$  by the monomials with at most one occurrence of  $T_n$ .

The proof will be by induction on  $n$ . Let  $M$  be a monomial in  $T_1, \dots, T_n$  with two occurrences of  $T_n$  at least. Displaying two consecutive occurrences of  $T_n$  in  $M$ , we write  $M = M_1 T_n M_2 T_n M_3$ , where we can assume that  $M_2$  is a monomial in  $T_1, \dots, T_{n-1}$  only. Assume by induction that  $M_2$  contains  $T_{n-1}$  at most once. If  $M_2$  does not contain  $T_{n-1}$  at all, then

$$M = M_1 M_2 T_n^2 M_3 = (q-1) M_1 M_2 T_n M_3 + q M_1 M_2 M_3,$$

reducing the number of occurrences of  $T_n$  in each new monomial. If  $M_2$  contains  $T_{n-1}$  exactly once, we can write  $M_2 = M' T_{n-1} M''$ , with  $M'$ ,  $M''$  monomials in  $T_1, \dots, T_{n-2}$  and then,

$$M = M_1 M' T_n T_{n-1} T_n M'' M_3,$$

using the fact that  $T_1, \dots, T_{n-2}$  commute with  $T_n$ . But now,  $T_n T_{n-1} T_n = T_{n-1} T_n T_{n-1}$  yields

$$M = M_1 M' T_{n-1} T_n T_{n-1} M'' M_3,$$

reducing again the number of occurrences of  $T_n$ .

Hence, every element of  $H_{n+1}$  is a sum  $a + \sum_i b_i T_n c_i$  with  $a, b_i, c_i$  coming from  $H_n$  and it is now clear that  $\varphi$  is surjective.

**CLAIM 3.** *Monomials in normal form generate  $H_{n+1}$  over  $K$ .*

We have actually proved a little more than was stated in Claim 2. Consider the following lists of monomials:

$$S_1 = \{1, T_1\},$$

$$S_2 = \{1, T_2, T_2 T_1\},$$

$$S_3 = \{1, T_3, T_3 T_2, T_3 T_2 T_1\},$$

...

$$S_i = \{1, T_i, T_i T_{i-1}, \dots, T_i T_{i-1} \dots T_1\},$$

...

$$S_n = \{1, T_n, T_n T_{n-1}, \dots, T_n T_{n-1} \dots T_1\}.$$

Note the property that  $V_i \in S_i$  implies  $T_{i+1} V_i \in S_{i+1}$ .

Consider the set of monomials  $M = U_1 \cdot U_2 \cdot \dots \cdot U_n$  for all possible choices of  $U_i \in S_i$ ,  $i = 1, \dots, n$ . We shall say that these monomials are in *normal form*. There are  $(n+1)!$  of them.

We claim that these monomials  $M$  generate  $H_{n+1}$  as a  $K$ -space. Consequently,  $\dim_K H_{n+1} \leq (n+1)!$  and also  $\dim_K \{H_n \oplus H_n \otimes H_n\} \leq (n+1)!$ , where the tensor product is over  $H_{n-1}$  as above.

*Proof.* We may assume by *induction* that the claim holds for  $H_n$ . As  $H_{n+1}$  is generated over  $K$  by monomials  $M_0$  and  $M = M_1 T_n M_2$ , where  $M_0, M_1, M_2$  are monomials in  $T_1, \dots, T_{n-1}$ , and as the induction hypothesis makes the case of  $M_0$  clear, we concentrate on  $M = M_1 T_n M_2$ . By induction,  $M_2$  is a  $K$ -linear combination of monomials of the form  $V_1 \cdot V_2 \cdot \dots \cdot V_{n-1}$ , with  $V_i \in S_i$  for  $i = 1, \dots, n-1$ . We have

$$M_1 T_n V_1 V_2 \dots V_{n-1} = M'_1 T_n V_{n-1} = M'_1 U_n,$$

with  $U_n = T_n V_{n-1} \in S_n$ . By induction again,  $M'_1$  is a  $K$ -linear combination of monomials of the form  $U_1 \cdot U_2 \cdot \dots \cdot U_{n-1}$  with  $U_i \in S_i$ . Thus  $M$  is a  $K$ -linear combination of monomials  $U_1 \cdot U_2 \cdot \dots \cdot U_n$  as desired and  $\dim_K H_{n+1} \leq (n+1)!$ .

This shows also that  $H_n \otimes_{H_{n-1}} H_n$  is spanned over  $K$  by the subspaces  $H_n \otimes U_{n-1}$  with  $U_{n-1} \in S_{n-1}$ . Therefore, its  $K$ -dimension is at most  $n! \cdot n$ , so that the proof of claim 3 is complete.

*Remark.* Let  $\mathfrak{S}_{n+1}$  be the symmetric group on  $\{1, \dots, n+1\}$ , and denote by  $s_i$  the transposition  $(i, i+1)$ . The same argument as above shows that any  $\pi \in \mathfrak{S}_{n+1}$  can be written as a product  $w_1 \cdot w_2 \dots \cdot w_n$ , where

$$w_i \in \{1, s_i, s_i s_{i-1}, \dots, s_i s_{i-1} \dots s_1\}.$$

We shall use this remark presently in the proof of the following claim 4.

*Exercise.* Deduce from the remark that  $\mathfrak{S}_{n+1}$  has a presentation on generators  $s_1, \dots, s_n$  with the relations

$$\begin{aligned} s_i s_j &= s_j s_i \quad \text{whenever } |i - j| \geq 2 \quad \text{with } i, j = 1, \dots, n, \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} \quad \text{for } i = 1, \dots, n-1, \\ s_i^2 &= 1 \quad \text{for } i = 1, \dots, n. \end{aligned}$$

**CLAIM 4.** *The monomials in normal form  $M = U_1 \cdot U_2 \cdot \dots \cdot U_n$ , with  $U_i \in S_i$  for  $i = 1, \dots, n$  are  $K$ -linearly independent. Also, the map  $\phi$  is an isomorphism.*

*Proof.* Denote by  $l: \mathfrak{S}_{n+1} \rightarrow N$  the word length in  $\mathfrak{S}_{n+1}$ , relative to the generators  $\{s_1, s_2, \dots, s_n\}$ . For  $i \in \{1, \dots, n\}$ , define  $L_i \in \text{End}_K(K\mathfrak{S}_{n+1})$  by

$$L_i(\pi) = \begin{cases} s_i\pi & \text{if } l(s_i\pi) > l(\pi), \\ qs_i\pi + (q-1)\pi & \text{if } l(s_i\pi) < l(\pi), \end{cases}$$

for every  $\pi \in \mathfrak{S}_{n+1}$ .

The crucial fact is the following

ASSERTION. *There is an algebra map  $L: H_{n+1} \rightarrow \text{End}_K(K\mathfrak{S}_{n+1})$  such that  $L(T_i) = L_i$  for  $i = 1, \dots, n$ .*

To prove the assertion, we have to check that the endomorphisms  $L_i \in \text{End}_K(K\mathfrak{S}_{n+1})$  satisfy the defining relations of the Hecke algebra  $H_{n+1}$ . For this, see the following three claims.

Assuming the assertion, consider a monomial in normal form  $M = U_1 \cdot U_2 \dots \cdot U_n$  as above. Then,  $L(M)$  maps  $1 \in K\mathfrak{S}_{n+1}$  to  $w_1 \cdot w_2 \dots \cdot w_n$ , where  $w_i = s_i s_{i-1} \dots s_{i-j}$  if  $U_i = T_i T_{i-1} \dots T_{i-j}$ . The remark after claim 3 now shows that any of the  $(n+1)!$  elements of  $\mathfrak{S}_{n+1}$  is of the form  $w_1 \cdot w_2 \dots \cdot w_n$ , so that these elements are  $K$ -linearly independent in  $K\mathfrak{S}_{n+1}$ . But, as the map from  $H_{n+1}$  to  $K\mathfrak{S}_{n+1}$  which sends  $x$  to  $L(x)(1)$  is  $K$ -linear, this implies that the elements  $M = U_1 \cdot U_2 \dots \cdot U_n$  in normal form must also be linearly independent. Hence,  $\dim_K H_{n+1} = (n+1)!$ .

Now, a dimension count shows that the surjective map  $\varphi$  is an isomorphism.

It remains to prove the above assertion: The  $L_i$ 's satisfy the defining relations for  $H_{n+1}$ .

CLAIM 5.  $L_i^2 = (q-1)L_i + q$  for  $i = 1, \dots, n$ .

*Proof.* Let  $\pi \in \mathfrak{S}_{n+1}$ . If  $l(s_i\pi) > l(\pi)$ , then

$$\begin{aligned} L_i^2(\pi) &= L_i(s_i\pi) = qs_i^2\pi + (q-1)s_i\pi \\ &= (q-1)s_i\pi + q\pi = ((q-1)L_i + q)(\pi). \end{aligned}$$

If on the other hand,  $l(s_i\pi) < l(\pi)$ , set  $\pi' = s_i\pi$  and observe that  $l(s_i\pi') > l(\pi')$ . Thus,

$$\begin{aligned} L_i^2(\pi) &= L_i(qs_i\pi + (q-1)\pi) = L_i(q\pi' + (q-1)\pi) \\ &= qs_i\pi' + (q-1)L_i(\pi) = ((q-1)L_i + q)(\pi). \end{aligned}$$

The next claim will be used in proving the last two types of relations for the endomorphisms  $L_i$ .

CLAIM 6. For  $j = 1, \dots, n$  define  $R_j \in \text{End}_K(K\mathfrak{S}_{n+1})$  by

$$R_j(\pi) = \begin{cases} \pi s_j & \text{if } l(\pi s_j) > l(\pi), \\ q\pi s_j + (q-1)\pi & \text{if } l(\pi s_j) < l(\pi). \end{cases}$$

Then,  $L_i R_j = R_j L_i$  for all  $i, j \in \{1, \dots, n\}$ .

*Proof.* Choose  $i, j \in \{1, \dots, n\}$  and  $\pi \in \mathfrak{S}_{n+1}$ . The proof that  $L_i R_j(\pi) = R_j L_i(\pi)$  is by direct verification from the definitions of the operators  $L_i, R_j$  and is divided into six cases.

$$(6.1) \quad l(s_i \pi s_j) = l(\pi) + 2,$$

$$(6.2) \quad l(s_i \pi s_j) = l(\pi) - 2,$$

$$(6.3)-(6.6) \quad l(s_i \pi s_j) = l(\pi) \quad \text{and}$$

$$l(s_i \pi) = l(\pi) + \varepsilon, \quad \text{where } \varepsilon = \pm 1,$$

$$l(\pi s_j) = l(\pi) + \varepsilon', \quad \text{where } \varepsilon' = \pm 1.$$

The first two cases are straightforward calculations.

Among the last four cases, two are also trivial, namely those with  $\varepsilon \neq \varepsilon'$ . There remain the two cases with  $\varepsilon = \varepsilon' = \pm 1$ . Then, the *exchange lemma* applied to the symmetric group viewed as a Coxeter group (on the generators  $s_1, \dots, s_n$ ) implies that in these cases we have  $s_i \pi = \pi s_j$ . (If  $\varepsilon = \varepsilon' = +1$ , this equality is given as property C in Bourbaki, Groupes et Algèbres de Lie, Chap. IV, n° 1.7. If  $\varepsilon = \varepsilon' = -1$ , the same property yields  $s_i(\pi s_j) = (\pi s_j)s_j$ .) This is just what is needed to complete the verification of  $L_i R_j(\pi) = R_j L_i(\pi)$ .

CLAIM 7.  $L_i L_j = L_j L_i$  whenever  $|i-j| \geq 2$ ,

$$L_i L_{i+1} L_i = L_{i+1} L_i L_{i+1}.$$

*Proof.* Let  $\pi \in \mathfrak{S}_{n+1}$ . Write  $\pi = s_{i_1} \cdot s_{i_2} \dots \cdot s_{i_r}$  in reduced form, i.e. with  $r = l(\pi)$ . We thus have  $\pi = R_{i_r} R_{i_{r-1}} \dots R_{i_1}(1)$ .

Setting  $R = R_{i_r} \dots R_{i_1}$ , we have

$$\begin{aligned} L_i L_j(\pi) &= L_i L_j R(1) = R L_i L_j(1) \quad \text{by claim 6,} \\ &= R(s_i s_j) = R(s_j s_i) \quad \text{since } |i-j| \geq 2, \quad \text{and thus} \end{aligned}$$

$$L_i L_j(\pi) = L_j L_i(\pi).$$

Since this holds for every  $\pi \in \mathfrak{S}_{n+1}$ , one has  $L_i L_j = L_j L_i$ .

A similar calculation, based on the same principle, proves that  $L_i L_{i+1} L_i = L_{i+1} L_i L_{i+1}$  for  $i = 1, \dots, n-1$ .

This completes the proof of Proposition 4.1.