§1. Introduction and historical remarks

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **32 (1986)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **26.09.2024**

§ 1. Introduction and historical remarks

Knot theory was born around the year 1867 in Scotland from the imagination of three physicists; two Scotsmen living in Edinburgh: J. C. Maxwell and P. G. Tait and one Irishman living in Glasgow: W. Thomson (Lord Kelvin). For more details, see [Kn].

The Transactions of the Royal Philosophical Society of Edinburgh provide ample testimony of the dedication and enthusiasm of these pioneers, trying to understand the structure of matter before quantum theory was invented, and knot theory without topological invariants.

According to Thomson's theory of vortex atoms, the chemical elements are constituted by small knots formed by the vortex lines of ether. For physical reasons, these knots have to be "kinetically stable", as Thomson and Tait said. In their opinion, this condition was going to prevent many knots from giving rise to vortex atoms.

Having this in mind, Tait embarked on a quite formidable program:

(1) Try to classify knots in 3-space;

(2) Try to establish a hierarchy among knots, relying on some notion of complexity;

(3) Understand why many of the simple knots cannot occur in vortex atoms (due to the stability condition).

In Tait's paper, this last point is stated as one of the main problems of the whole subject.

(4) Explain the position of the lines in the spectrum of a chemical element from the shape of the corresponding knot.

From an epistemological point of view, this program is remarkable: Thomson and Tait (T and T' as their friends used to call them) are looking for very complicated mathematical objects, in contrast with the attitude of many scientists trying to find a simple mathematical model when they attempt to explain a new area in the natural sciences.

If one reads between the lines in Tait's paper, one can guess that he started working on (1) and (2) full of the hope that it should not be too difficult. However, he was aware of the fact that he was opening an entirely new field and that surprises might well show up. Later on, he confessed that the subject was harder than he had expected...

During the elaboration of Tait's first paper, Maxwell told him about the work on knots by C. F. Gauss and J. B. Listing who had somewhat anticipated Tait's starting point: knot projections, alternating knots, chessboard.

As to Maxwell, his interest for knots came from his theory of electromagnetism. For instance, he gave in [Ma] a lovely interpretation of Gauss integral formula for the linking coefficient of two knots in 3-space: it is (up to a factor) equal to the work required to move a magnet pole along one knot while the other knot is run by an electric current. This interpretation is repeated by Tait in [Tai]. One can see, along the way, Seifert surfaces being introduced by Tait via the following physical argument: If one has an orientable surface Σ in 3-space whose boundary is a given knot, and if one "magnetizes the surface normally and constantly", as Tait says, then the work required to move a magnet pole on another knot will be the same as if the boundary of Σ were run by an electric current. Tait thus uses the 2-chain given by Σ to compute linking coefficients.

Note. Today, G. de Rham himself says that he chose the terminology "courant" for similar reasons. The "courants", like homology, are dual to cohomology and one can think of 1-dimensional cycles as electric currents.

Tait thought of a knot as being a rubber band in everyday 3-dimensional space. Two positions of the band represent the same knot if one can deform one position of the band into the other. In modern terms, this is non-oriented ambient isotopy.

To measure the complexity of a knot, Tait introduced what he called the (degree of) knottiness. This is called today the crossing number of the knot. By definition, it is the minimal number of double points among all projections of the knot. We shall use the notation $c(K)$.

Tait also introduced the beknottedness, now called unknotting number. He did not use it very much to measure knot complexity because he soon realized that its determination was difficult. We shall not talk about this invariant in this paper, although the second integer which appears in the inductive proof of uniqueness of the polynomial $P_K(l, m)$ in § 3 is clearly related to it.

Tait's papers contain few proofs which are acceptable by the standards of 20-th century topology. They rely on principles, not always very explicitly stated, which seemed obvious to the author, but which are in fact unproved statements. Nowadays, knot theorists have more or less agreed on the meaning of these principles and have summarized them under the name of "Tait conjectures". They are all related to the minimal crossing number of a knot. (See § 9 in this paper.)

A paradox in the achievements of 3-dimensional topology between 1965 and 1985 is the following: Knot theory was gradually embodied in the more general theory of 3-dimensional manifolds. Classifications were attempted, and sometimes attained by using very refined geometrical tools such as the Waldhausen-Jaco-Shalen-Johanson theory on the embeddings of Seifert manifolds in a Haken manifold. And yet, these refined methods could not cope with simple questions related to knot projections. In fact, during this period, the old time point of view, using projections, was almost forgotten (except by a few people, for instance John Conway).

Today, Jones polynomials and more precisely L. Kauffman's very clever and very elementary way of looking at the one-variable polynomial $V_K(t)$ have put again knot projections under the spot-light. The one-variable polynomial is the main ingredient in the proofs of several of Tait's conjectures which have remained unproved for more than a century.

This paper is devoted to a presentation of these recent achievements, mainly due to V. Jones, L. Kauffman and K. Murasugi.

We shall give the definition and prove some of the properties of the two-variable Laurent polynomial $P(K) \in \mathbf{Z}[l, l^{-1}, m, m^{-1}]$ associated with every oriented link K. The approach chosen here is that of V. Jones and A. Ocneanu. Another approach which uses the notion of skein invariance is due independently to many mathematicians: P. Freyd, D. Yetter, J. Hoste, W. Lickorish, K. Millett, J. Prztycki and P. Traczyk. Although we do discuss skein invariance in this paper, we do not go into the question of using it to define the polynomial $P(K)$.

As many mathematicians have worked simultaneously on various aspects of the definition of the polynomial, it is difficult to give proper credit to everyone. We apologize in advance for any missing ascription. We hope all will agree that V. Jones has been the one pioneer who got the subject started.

§ 2. Link diagrams

A link K in S^3 (or R^3) is a 1-dimensional compact smooth manifold without boundary. We shall use $r = r(K)$ for the number of components of K. A knot is a link with one component.

Most of the time K will be oriented.

Two oriented links K, K' are *ambient isotopic* if there exists a diffeomorphism $h: S^3 \to S^3$ of degree $+1$, such that $h(K) = K'$ and $h_{|K}$ is also of degree $+1$ on each component.