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$$b_{x_1}\lambda_{f_1}b_{y_2}\lambda_{f'_2}\ldots b_{y_s}\lambda_{f'_s}b_v R$$
$$= a_{x_1}\lambda_{f_1}\lambda_{\bar{f}_1}a_{x_1}^{-1}b_{x_1}\lambda_{f_1}b_{y_2}\lambda_{f'_2}\ldots b_{y_s}\lambda_{f'_s}b_v R$$
$$= a_{x_1}\lambda_{f_1}\lambda_{\bar{f}_1}c_{x_1}\lambda_{f_1}b_{y_2}\lambda_{f'_2}\ldots b_{y_s}\lambda_{f'_s}b_v R$$
$$= a_{x_1}\lambda_{f_1}(\gamma_{\bar{f}_1}c\gamma_{f_1})_{z_1}b_{y_2}\lambda_{f'_2}\ldots b_{y_s}\lambda_{f'_s}b_v R$$
$$= a_{x_1}\lambda_{f_1}(\gamma_{\bar{f}_1}c\gamma_{f_1})_{x_2}b_{y_2}\lambda_{f'_2}\ldots b_{y_s}\lambda_{f'_s}b_v R$$
$$= a_{x_1}\lambda_{f_1}a'_{x_2}b_{y_2}\lambda_{f'_2}\ldots b_{y_s}\lambda_{f'_s}b_v R$$

where $a'_{x_2} = (\gamma_{\bar{f}_1}c\gamma_{f_1})_{x_2}$. If x_2 happens to equal y_2 then we simplify this further to

$$a_{x_1}\lambda_{f_1}a''_{x_2}\lambda_{f_2}b_{y_3}\lambda_{f'_3}\ldots b_{y_s}\lambda_{f'_s}b_v R$$

where a''_{x_2} is the product $a'_{x_2}b_{y_2}$ in G_{x_2}. We now compare a_{x_2} with a'_{x_2} if $x_2 \neq y_2$, noting that $\gamma_{f_2} = 1$ in this case, or with a''_{x_2} if $x_2 = y_2$, and repeat the process. Eventually we obtain

$$b_{x_1}\lambda_{f_1}b_{y_2}\lambda_{f'_2}\ldots b_{y_s}\lambda_{f'_s}b_v R = a_{x_1}\lambda_{f_1}a_{x_2}\lambda_{f_2}\ldots a_{x_r}\lambda_{f_r}a''_v R\,.$$

As $g = a_{x_1}\gamma_{f_1}\ldots a_{x_r}\gamma_{f_r}a_v = a_{x_1}\gamma_{f_1}\ldots a_{x_r}\gamma_{f_r}a''_v$ we see that $a''_v = a_v$. This completes the proof that ψ is well defined.

5. Nearest fixed points

To show ψ is a homomorphism we shall verify

$$\psi(hg) = \psi(h)\psi(g)$$

under the assumption that h either leaves some vertex of T fixed or is one of the elements γ_f. This is sufficient because the elements of the G_w (w a vertex of T) together with the γ_f (f an edge of $X/G-M$) form a set of generators for G.

Suppose h fixes the vertex w of T. Walk along the geodesic $\overrightarrow{vw}$ and let x be the first vertex we meet which is left fixed by h. Then $\overrightarrow{vx}$ is contained in T, and $\overrightarrow{vx}$ followed by $h(\overleftarrow{xv})$ is the geodesic from v to hv. This quits T for the first time at x and we see that

$$\psi(h) = h_x R\,.$$

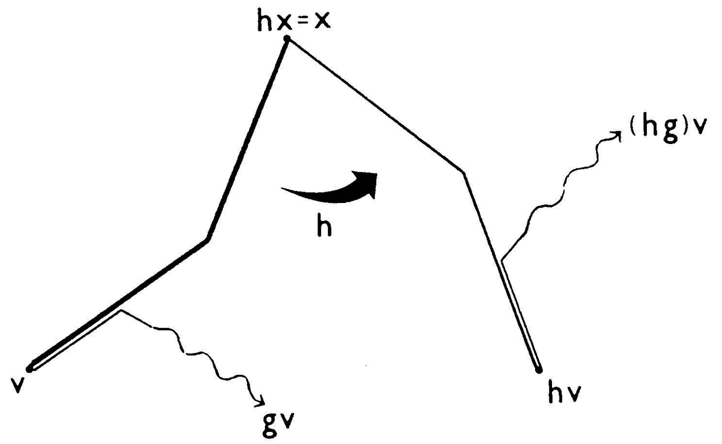

FIGURE 3

Using the geodesic from v to gv we have $\psi(g) = a_{x_1} \lambda_{f_1} \dots a_{x_r} \lambda_{f_r} a_v R$ in the usual way. Therefore

$$\psi(h)\psi(g) = h_x a_{x_1} \lambda_{f_1} \dots a_{x_r} \lambda_{f_r} a_v R .$$

In order to compute $\psi(hg)$ we need the geodesic from v to $(hg)v$. We can construct this as follows, take $\overrightarrow{v\,hv}$ followed by the image of $\overrightarrow{v\,gv}$ under h and remove any round trips.

If $\overrightarrow{v\,gv}$ does not contain all of $\overrightarrow{vx}$ (Figure 3) then $\overrightarrow{v(hg)v}$ leaves T for the first time at x. A tail wag of $\overrightarrow{v(hg)v}$ using h_x^{-1} leads us to a path which has the same tail as $\overrightarrow{v\,gv}$, then the process continues as for g. Thus

$$\psi(hg) = h_x a_{x_1} \lambda_{f_1} \dots a_{x_r} \lambda_{f_r} a_v R = \psi(h)\psi(g) .$$

Otherwise $\overrightarrow{v\,gv}$ contains all of $\overrightarrow{vx}$ (Figure 4) and we split the argument into three cases.

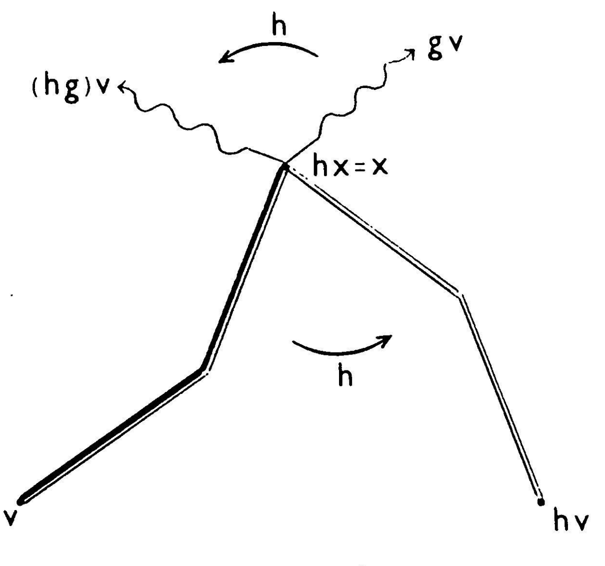

FIGURE 4

(a) $\overrightarrow{v\,gv}$ stays in T for at least one more edge after x. Then $\overrightarrow{v(hg)v}$ must leave T at x. As above, a first choice of h_x^{-1} leads to a path with the same tail as $\overrightarrow{v\,gv}$.

(b) $\overrightarrow{v\,gv}$ and $\overrightarrow{v(hg)v}$ both leave T at x. Then $x_1 = x$ and we write a_x instead of a_{x_1}. A first tail wag of $\overrightarrow{v(hg)v}$ using $\gamma_{\bar{f}_1}(h_x a_x)^{-1}$ produces the same path as a first tail wag of $\overrightarrow{v\,gv}$ using $\gamma_{\bar{f}_1} a_x^{-1}$. Thus

$$\psi(hg) = h_x a_x \lambda_{f_1} a_{x_2} \lambda_{f_2} \dots a_{x_r} \lambda_{f_r} a_v R = \psi(h)\psi(g) .$$

(c) $\overrightarrow{v\,gv}$ leaves T at x, but $\overrightarrow{v(hg)v}$ stays in T for at least one more edge after x. Then $x_1 = x$, $\gamma_{f_1} = 1$ and we may as well equate a_{x_1} with h_x^{-1}. A first tail wag of $\overrightarrow{v\,gv}$ using h_x gives a path with the same tail as $\overrightarrow{v(hg)v}$. Thus

$$\begin{aligned}\psi(hg) &= a_{x_2} \lambda_{f_2} \dots a_{x_r} \lambda_{f_r} a_v R \\ &= h_x h_x^{-1} a_{x_2} \lambda_{f_2} \dots a_{x_r} \lambda_{f_r} a_v R \\ &= \psi(h)\psi(g) .\end{aligned}$$

Suppose finally that $h = \gamma_f$ for some edge f of $X/G - M$. As usual e is the chosen lift of f into X with $x = i(e) \in T$ and $z = t(\gamma_{\bar{f}}\, e)$. Let $y = i(\gamma_{\bar{f}}\, e)$. The geodesic from v to $\gamma_f v$ is made up of $\overrightarrow{vx}$ followed by e followed by $\gamma_f(\overrightarrow{zv})$. This leaves T for the first time at x and a single tail wag using $\gamma_{\bar{f}}$ produces $\overrightarrow{zv}$. Therefore

$$\psi(\gamma_f) = \lambda_f R .$$

To obtain the geodesic from v to $(\gamma_f g)v$ we follow $\overrightarrow{v\gamma_f v}$ by $\gamma_f(\overrightarrow{v\,gv})$ and then remove any round trips (Figure 5). If $\overrightarrow{v\,gv}$ does not contain $\overrightarrow{vy}$, then $\overrightarrow{v(\gamma_f g)v}$ leaves T for the first time at x and a single tail wag using $\gamma_{\bar{f}}$

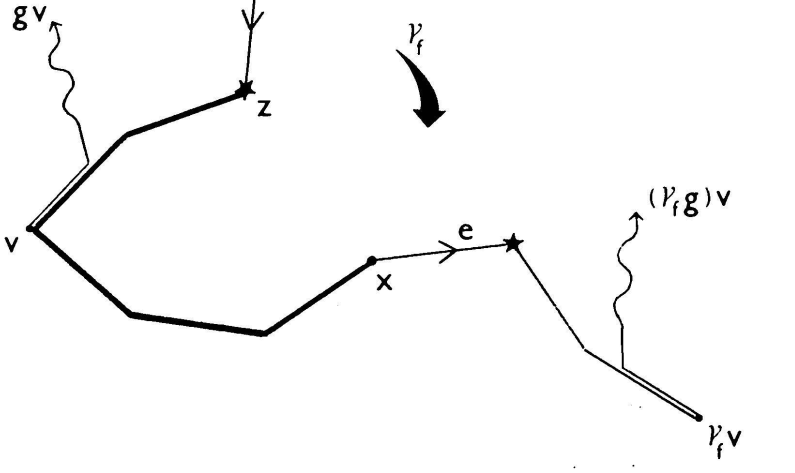

FIGURE 5

produces a path with the same tail as $\overrightarrow{v\,gv}$. The process then continues as for g and

$$\psi(\gamma_f g) = \lambda_f\, a_{x_1}\, \lambda_{f_1} \dots a_{x_r}\, \lambda_{f_r}\, a_v R = \psi(\gamma_f)\psi(g)\,.$$

Otherwise $\overrightarrow{v\,gv}$ contains $\overrightarrow{vy}$. Then $x_1 = z$, $\gamma_{f_1} \doteq \gamma_{\bar{f}}$ and we may as well take $a_{x_1} = 1$. A first tail wag of $\overrightarrow{v\,gv}$ using γ_f leaves a path with the same tail as $\overrightarrow{v(\gamma_f g)v}$. Thus

$$\begin{aligned} \psi(\gamma_f g) &= a_{x_2}\, \lambda_{f_2} \dots a_{x_r}\, \lambda_{f_r}\, a_v R \\ &= \lambda_f\, \lambda_{\bar{f}}\, a_{x_2}\, \lambda_{f_2} \dots a_{x_r}\, \lambda_{f_r}\, a_v R \\ &= \psi(\gamma_f)\psi(g)\,. \end{aligned}$$

This completes the proof that ψ is a homomorphism.

Our construction of ψ ensures that if $\psi(g) = R$ then $g = 1$. So ψ is injective. The cosets $h_w R$ (w a vertex of T and $h(w) = w$) and $\lambda_f R$ (f an edge of X/G) together generate $[(*\, G_w)*F]/R$. Now $\psi(h) = h_x R$ where x is the nearest fixed point of h to v. But h fixes all of $\overrightarrow{xw}$ so

$$\psi(h) = h_x R = h_w R\,.$$

Also

$$\psi(\gamma_f) = \lambda_f R\,.$$

Therefore the image of ψ is all of $[(*\, G_w)*F]/R$ and we have shown that ψ is an isomorphism.

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