

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 32 (1986)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** TREES, TAIL WAGGING AND GROUP PRESENTATIONS  
**Autor:** Armstrong, M. A.  
**Kapitel:** 3. Tail wagging  
**DOI:** <https://doi.org/10.5169/seals-55090>

#### Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

#### Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

#### Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 18.08.2025

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

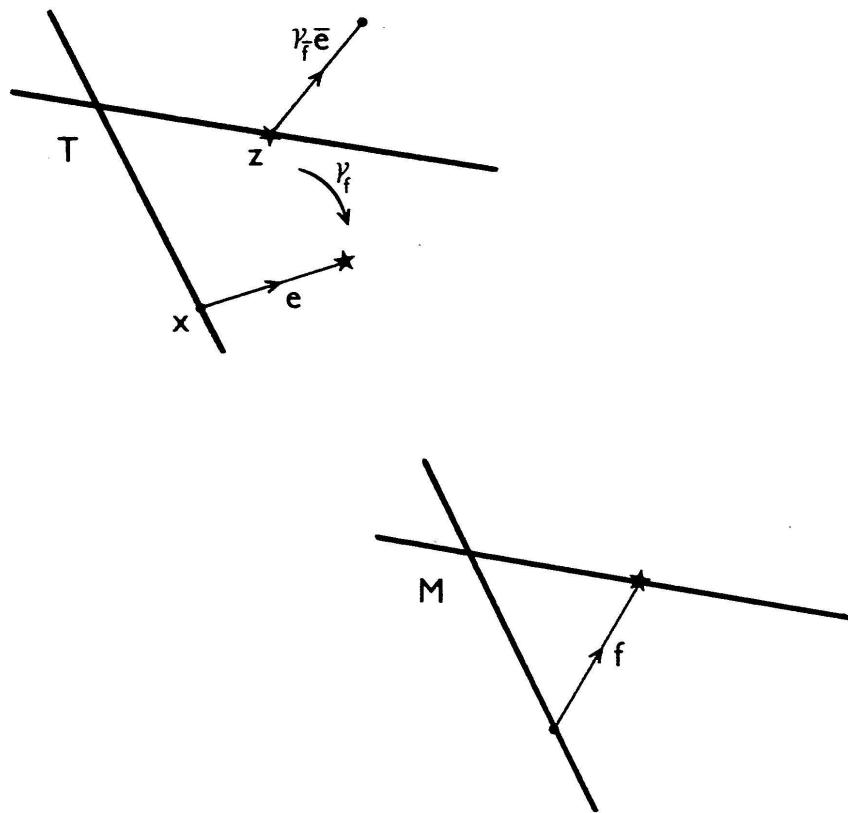


FIGURE 1

## 3. TAIL WAGGING

With the notation established above let  $*G_w$  denote the free product of the stabilizers of the vertices of  $T$ , and  $F$  the free group generated by symbols  $\lambda_f$ , one for each edge  $f$  of  $X/G$ . Let  $R$  be the normal consequence in  $(*G_w)*F$  of the words

$$\begin{aligned}\lambda_f & \quad (f \text{ an edge of } M), \\ \lambda_{\bar{f}} \lambda_f & \quad \text{and} \\ \lambda_{\bar{f}} g_x \lambda_f (\gamma_{\bar{f}} g \gamma_f)^{-1} &\end{aligned}$$

We shall produce an isomorphism

$$\psi: G \rightarrow [(*G_w)*F]/R.$$

Choose a vertex  $v$  of  $T$  as base point. If  $g \in G$  fixes  $v$  set

$$\psi(g) = g_v R$$

where as usual  $g_v$  is the element  $g$  interpreted as a member of  $G_v$ . If  $g$  moves  $v$  then it sends it outside  $T$  because no two vertices of  $T$  lie in the same orbit. Let  $e_1 e_2 \dots e_n$  be the geodesic which joins  $v$  to  $gv$  and suppose  $e_m$  is the first edge that is *not* in  $T$ . The path  $e_m e_{m+1} \dots e_n$  will be called the *tail* of  $\overrightarrow{v gv}$ . Let  $x_1$  be the initial vertex of  $e_m$ . Project  $e_m$  into  $X/G$  to give an edge  $f_1$ . The canonical lift  $e^1$  of  $f_1$  into  $X$  has its initial vertex in  $T$ , so  $i(e^1) = x_1$ . Choose an element  $a_{x_1} \in G_{x_1}$  which sends  $e^1$  to  $e_m$ . Let

$$e_k^1 = (\gamma_{f_1} a_{x_1}^{-1}) e_k$$

for  $m+1 \leq k \leq n$ , and replace  $e_1 e_2 \dots e_n$  by the new path  $e_{m+1}^1 e_{m+2}^1 \dots e_n^1$ . We call this process *tail wagging*. Our new path begins at

$$z_1 = t(\gamma_{f_1} e^1) = i(e_{m+1}^1)$$

which is a vertex of  $T$  and ends at  $(\gamma_{f_1} a_{x_1}^{-1} g)v$ , see Figure 2. We walk along it to the first point  $x_2$  where it quits  $T$  and repeat the above

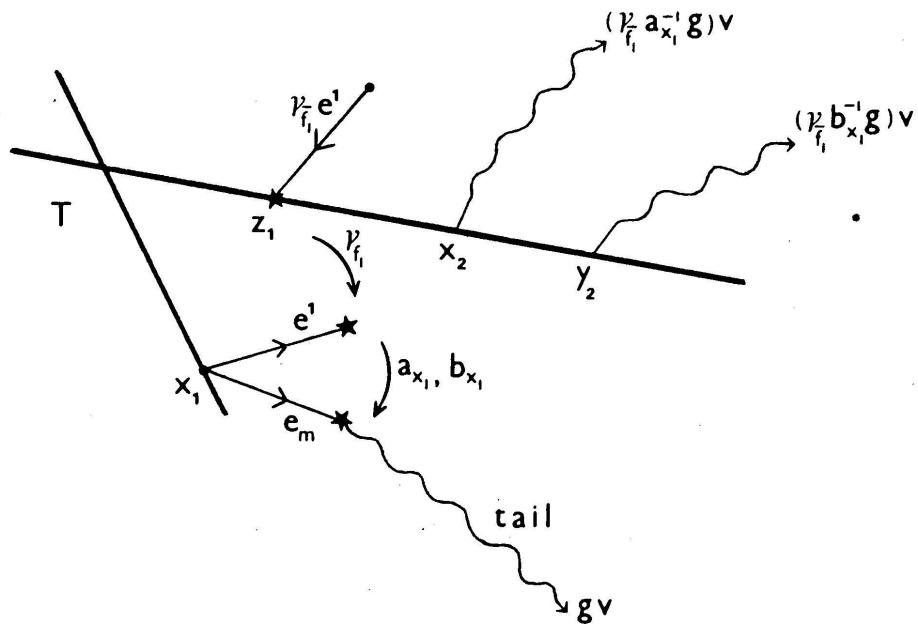


FIGURE 2

procedure. Since we shorten the tail at each step we eventually obtain a path which lies entirely in  $T$  and ends at say

$$(\gamma_{\bar{f}_r} a_{x_r}^{-1} \dots \gamma_{\bar{f}_2} a_{x_2}^{-1} \gamma_{\bar{f}_1} a_{x_1}^{-1} g)v.$$

Then  $\gamma_{\bar{f}_r} a_{x_r}^{-1} \dots \gamma_{\bar{f}_1} a_{x_1}^{-1} g$  must fix  $v$ , say  $\gamma_{\bar{f}_r} a_{x_r}^{-1} \dots \gamma_{\bar{f}_1} a_{x_1}^{-1} g = a_v \in G_v$ . We now have

$$g = a_{x_1} \gamma_{f_1} \dots a_{x_r} \gamma_{f_r} a_v$$

and we somewhat optimistically define

$$\psi(g) = a_{x_1} \lambda_{f_1} \dots a_{x_r} \lambda_{f_r} a_v R.$$

#### 4. AN INEFFICIENT CHOICE

Is  $\psi$  well defined? The geodesic from  $v$  to  $gv$  is certainly unique, as is the first point  $x_1$  where it leaves  $T$  and its first edge  $e_m$  outside  $T$ . Both the edge  $e^1$  and the group element  $\gamma_{f_1}$  are now determined by our original construction. The only ambiguity at this stage is the choice of the element  $a_{x_1} \in G_{x_1}$  which maps  $e^1$  to  $e_m$ . A different choice  $b_{x_1}$  will give a path from  $z_1$  to  $(\gamma_{\bar{f}_1} b_{x_1}^{-1} g)v$  which leaves  $T$  for the first time at say  $y_2$ . The first edge outside  $T$  will project to an edge  $f'_2$  of  $X/G$  and so on until eventually we have  $g$  expressed as

$$g = b_{x_1} \gamma_{f_1} b_{y_2} \gamma_{f'_2} \dots b_{y_s} \gamma_{f'_s} b_v.$$

We must show that  $a_{x_1} \lambda_{f_1} a_{x_2} \lambda_{f_2} \dots a_{x_r} \lambda_{f_r} a_v$  and  $b_{x_1} \lambda_{f_1} b_{y_2} \lambda_{f'_2} \dots b_{y_s} \lambda_{f'_s} b_v$  determine the same left coset of  $R$  in  $(*G_w)*F$ .

Agree to select  $a_{x_1}$  from  $G_{x_1}$  so that the tail of the resulting path is as long as possible. Continue in this way selecting  $a_{x_2}, a_{x_3} \dots$  so as to maximise the length of the tail at each stage. We shall compare any other set of choices with this rather inefficient selection.

Both  $a_{x_1}$  and  $b_{x_1}$  map  $e^1$  to  $e_m$ , so  $c = a_{x_1}^{-1} b_{x_1}$  must fix  $e^1$ . Also, due to our particular selection of  $a_{x_1}$ , the geodesic from  $z_1$  to  $x_2$  is left fixed by  $\gamma_{\bar{f}_1} c \gamma_{f_1}$ . Therefore