

# 1. Graphs

Objektyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **32 (1986)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **21.09.2024**

## **Nutzungsbedingungen**

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

## **Haftungsausschluss**

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

## TREES, TAIL WAGGING AND GROUP PRESENTATIONS

by M. A. ARMSTRONG

The Bass-Serre theorem gives a presentation for a group of automorphisms of a tree. Like all good theorems it has attracted considerable attention and there are now several proofs available [4], [3], [1]. Our goal is a natural elementary proof which makes maximal use of the geometry of the tree.

### 1. GRAPHS

A graph  $X$  consists of two sets  $E$  (directed edges) and  $V$  (vertices) and two functions

$$\begin{aligned} E &\rightarrow E, & e &\mapsto \bar{e} \\ E &\rightarrow V \times V, & e &\mapsto (i(e), t(e)) \end{aligned}$$

which satisfy  $\bar{\bar{e}} = e$ ,  $\bar{e} \neq e$  and  $i(\bar{e}) = t(e)$  for each  $e \in E$ . The vertices  $i(e)$ ,  $t(e)$  are the initial and terminal vertices of the directed edge  $e$ , and  $\bar{e}$  is the reverse of  $e$ . Henceforth we refer to directed edges simply as edges.

A path in  $X$  joining vertex  $u$  to vertex  $v$  is an ordered string of edges  $e_1 e_2 \dots e_n$  such that  $i(e_1) = u$ ,  $i(e_{k+1}) = t(e_k)$  for  $1 \leq k \leq n-1$ , and  $t(e_n) = v$ . If  $v = u$  we have a circuit. A path of the form  $e\bar{e}$  is a *round trip* and a circuit which does not contain any round trips will be called a loop. If any two distinct vertices may be joined by a path then the graph is connected. A *tree* is a connected graph which does not contain any loops.

Let  $X$  be a tree. A path in  $X$  is a *geodesic* if it does not contain any round trips. Given distinct vertices  $u, v$  of  $X$  there is a *unique* geodesic  $\overrightarrow{uv}$  which joins  $u$  to  $v$ .

An action of a group  $G$  on a graph  $X$  is an action of  $G$  on  $E$  and on  $V$  such that  $g\bar{e} = \overline{ge}$ ,  $i(ge) = gi(e)$ ,  $t(ge) = gt(e)$  and  $ge \neq \bar{e}$  for each  $e \in E$ . Because group elements are not allowed to reverse edges we have a

quotient graph  $X/G$ . When  $G$  acts on  $X$  we shall often say that  $G$  is a *group of automorphisms of  $X$* .

We adopt the usual notation whereby  $G_x$  denotes the stabilizer of a vertex  $x$ . If  $g \in G$  happens to fix  $x$  we write  $g_x$  for the element  $g$  thought of as a member of  $G_x$ . Of course  $G_e$  denotes the stabilizer of the edge  $e$ . If  $x$  is a vertex of  $e$  then  $G_e$  is a subgroup of  $G_x$ .

Suppose  $G$  acts on a tree  $X$ . If  $g \in G$  fixes the vertices  $u, v$  then it must fix the whole geodesic  $\overrightarrow{uv}$ , since otherwise the image of  $\overrightarrow{uv}$  under  $g$  would be a second geodesic from  $u$  to  $v$ .

## 2. LIFTING EDGES

Let  $G$  be a group of automorphisms of a tree  $X$ . Choose a maximal tree  $M$  in  $X/G$  and lift it [4, Proposition I.14] to a subtree  $T$  of  $X$ . The vertices of  $T$  form a set of representatives for the action of  $G$  on the vertices of  $X$ . For each pair of edges  $f, \bar{f}$  from  $X/G - M$  select one, say  $f$ , and lift it to an edge  $e$  of  $X$  which has its initial vertex  $x$  in  $T$ . Exactly one vertex  $z$  of  $T$  lies in the same orbit as  $t(e)$  and we choose an element  $\gamma_f$  from  $G$  that maps  $z$  onto  $t(e)$ . We can now lift  $\bar{f}$  to  $(\gamma_f)^{-1}\bar{e}$ . This has its initial vertex  $z$  in  $T$  and  $\gamma_{\bar{f}} = (\gamma_f)^{-1}$  sends the vertex  $x$  of  $T$  to its terminal vertex (Figure 1). Finally we extend the correspondence  $f \rightarrow \gamma_f$  over the edges of  $M$  by setting  $\gamma_f = 1$  (the identity element of  $G$ ) whenever  $f \in M$ .

The *Bass-Serre theorem* [4, Theorem I.13] gives the following presentation for  $G$ .

(a) *Generators.* The elements of all the  $G_w$  where  $w$  is a vertex of  $T$  and the  $\gamma_f$  where  $f$  is an edge of  $X/G$ .

(b) *Relations.* The internal relations of each stabilizer  $G_w$  together with

$$\gamma_f = 1 \text{ if } f \text{ is an edge of } M,$$

$$\gamma_{\bar{f}} = (\gamma_f)^{-1} \text{ and}$$

$$\gamma_{\bar{f}} g_x \gamma_f = (\gamma_{\bar{f}} g \gamma_f)_z \text{ where } e \text{ is the chosen lift of } f \text{ and } g \in G_e.$$

(If  $f$  is an edge of  $M$  then  $z = t(e)$  and the final relation reduces to  $g_x = g_z$  whenever  $g \in G_e$ ).