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TREES, TAIL WAGGING AND GROUP PRESENTATIONS

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The Bass-Serre theorem gives a presentation for a group of automorphisms of a tree. Like all good theorems it has attracted considerable attention and there are now several proofs available [4], [3], [1]. Our goal is a natural elementary proof which makes maximal use of the geometry of the tree.

1. Graphs

A *graph* X consists of two sets E (directed edges) and V (vertices) and two functions

$$E \to E\,, \quad e \mapsto \bar{e}$$
$$E \to V \times V\,, \quad e \mapsto (i(e), t(e))$$

which satisfy $\bar{\bar{e}} = e$, $\bar{e} \neq e$ and $i(\bar{e}) = t(e)$ for each $e \in E$. The vertices $i(e)$, $t(e)$ are the initial and terminal vertices of the directed edge e, and $\bar{e}$ is the reverse of e. Henceforth we refer to directed edges simply as edges.

A path in X joining vertex u to vertex v is an ordered string of edges $e_1 e_2 \dots e_n$ such that $i(e_1) = u$, $i(e_{k+1}) = t(e_k)$ for $1 \leqslant k \leqslant n-1$, and $t(e_n) = v$. If $v = u$ we have a circuit. A path of the form $e\bar{e}$ is a *round trip* and a circuit which does not contain any round trips will be called a loop. If any two distinct vertices may be joined by a path then the graph is connected. A *tree* is a connected graph which does not contain any loops.

Let X be a tree. A path in X is a *geodesic* if it does not contain any round trips. Given distinct vertices u, v of X there is a *unique* geodesic $\overrightarrow{uv}$ which joins u to v.

An action of a group G on a graph X is an action of G on E and on V such that $g\bar{e} = \overline{ge}$, $i(ge) = gi(e)$, $t(ge) = gt(e)$ and $ge \neq \bar{e}$ for each $e \in E$. Because group elements are not allowed to reverse edges we have a quotient graph X/G. When G acts on X we shall often say that G is a *group of automorphisms of* X.

We adopt the usual notation whereby G_x denotes the stabilizer of a vertex x. If $g \in G$ happens to fix x we write g_x for the element g thought of as a member of G_x. Of course G_e denotes the stabilizer of the edge e. If x is a vertex of e then G_e is a subgroup of G_x.

Suppose G acts on a tree X. If $g \in G$ fixes the vertices u, v then it must fix the whole geodesic $\overrightarrow{uv}$, since otherwise the image of $\overrightarrow{uv}$ under g would be a second geodesic from u to v.

2. Lifting edges

Let G be a group of automorphisms of a tree X. Choose a maximal tree M in X/G and lift it [4, Proposition I.14] to a subtree T of X. The vertices of T form a set of representatives for the action of G on the vertices of X. For each pair of edges f, $\bar{f}$ from $X/G - M$ select one, say f, and lift it to an edge e of X which has its initial vertex x in T. Exactly one vertex z of T lies in the same orbit as $t(e)$ and we choose an element γ_f from G that maps z onto $t(e)$. We can now lift $\bar{f}$ to $(\gamma_f)^{-1}\bar{e}$. This has its initial vertex z in T and $\gamma_{\bar{f}} = (\gamma_f)^{-1}$ sends the vertex x of T to its terminal vertex (Figure 1). Finally we extend the correspondence $f \to \gamma_f$ over the edges of M by setting $\gamma_f = 1$ (the identity element of G) whenever $f \in M$.

The *Bass-Serre theorem* [4, Theorem I.13] gives the following presentation for G.

(a) *Generators.* The elements of all the G_w where w is a vertex of T and the γ_f where f is an edge of X/G.

(b) *Relations.* The internal relations of each stabilizer G_w together with

$$\gamma_f = 1 \text{ if } f \text{ is an edge of } M,$$

$$\gamma_{\bar{f}} = (\gamma_f)^{-1} \text{ and}$$

$$\gamma_{\bar{f}}\, g_x \gamma_f = (\gamma_{\bar{f}}\, g\, \gamma_f)_z \text{ where } e \text{ is the chosen lift of } f \text{ and } g \in G_e.$$

(If f is an edge of M then $z = t(e)$ and the final relation reduces to $g_x = g_z$ whenever $g \in G_e$).