# 1. Graphs

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# TREES, TAIL WAGGING AND GROUP PRESENTATIONS

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The Bass-Serre theorem gives a presentation for a group of automorphisms of a tree. Like all good theorems it has attracted considerable attention and there are now several proofs available [4], [3], [1]. Our goal is a natural elementary proof which makes maximal use of the geometry of the tree.

## 1. GRAPHS

A *graph* $X$ consists of two sets $E$ (directed edges) and $V$ (vertices) and two functions

$$E \to E, \quad e \mapsto \bar{e}$$

$$E \to V \times V, \quad e \mapsto (i(e), t(e))$$

which satisfy $\bar{\bar{e}} = e$, $\bar{e} \neq e$ and $i(\bar{e}) = t(e)$ for each $e \in E$. The vertices $i(e)$, $t(e)$ are the initial and terminal vertices of the directed edge $e$, and $\bar{e}$ is the reverse of $e$. Henceforth we refer to directed edges simply as edges.

A path in $X$ joining vertex $u$ to vertex $v$ is an ordered string of edges $e_1 e_2 \dots e_n$ such that $i(e_1) = u$, $i(e_{k+1}) = t(e_k)$ for $1 \leqslant k \leqslant n-1$, and $t(e_n) = v$. If $v = u$ we have a circuit. A path of the form $e\bar{e}$ is a *round trip* and a circuit which does not contain any round trips will be called a loop. If any two distinct vertices may be joined by a path then the graph is connected. A *tree* is a connected graph which does not contain any loops.

Let $X$ be a tree. A path in $X$ is a *geodesic* if it does not contain any round trips. Given distinct vertices $u, v$ of $X$ there is a *unique* geodesic $\overrightarrow{uv}$ which joins $u$ to $v$.

An action of a group $G$ on a graph $X$ is an action of $G$ on $E$ and on $V$ such that $g\bar{e} = \overline{ge}$, $i(ge) = gi(e)$, $t(ge) = gt(e)$ and $ge \neq \bar{e}$ for each $e \in E$. Because group elements are not allowed to reverse edges we have a quotient graph $X/G$. When $G$ acts on $X$ we shall often say that $G$ is a *group of automorphisms of* $X$.

We adopt the usual notation whereby $G_x$ denotes the stabilizer of a vertex $x$. If $g \in G$ happens to fix $x$ we write $g_x$ for the element $g$ thought of as a member of $G_x$. Of course $G_e$ denotes the stabilizer of the edge $e$. If $x$ is a vertex of $e$ then $G_e$ is a subgroup of $G_x$.

Suppose $G$ acts on a tree $X$. If $g \in G$ fixes the vertices $u$, $v$ then it must fix the whole geodesic $\overrightarrow{uv}$, since otherwise the image of $\overrightarrow{uv}$ under $g$ would be a second geodesic from $u$ to $v$.

## 2. Lifting edges

Let $G$ be a group of automorphisms of a tree $X$. Choose a maximal tree $M$ in $X/G$ and lift it [4, Proposition I.14] to a subtree $T$ of $X$. The vertices of $T$ form a set of representatives for the action of $G$ on the vertices of $X$. For each pair of edges $f$, $\bar{f}$ from $X/G - M$ select one, say $f$, and lift it to an edge $e$ of $X$ which has its initial vertex $x$ in $T$. Exactly one vertex $z$ of $T$ lies in the same orbit as $t(e)$ and we choose an element $\gamma_f$ from $G$ that maps $z$ onto $t(e)$. We can now lift $\bar{f}$ to $(\gamma_f)^{-1}\bar{e}$. This has its initial vertex $z$ in $T$ and $\gamma_{\bar{f}} = (\gamma_f)^{-1}$ sends the vertex $x$ of $T$ to its terminal vertex (Figure 1). Finally we extend the correspondence $f \to \gamma_f$ over the edges of $M$ by setting $\gamma_f = 1$ (the identity element of $G$) whenever $f \in M$.

The *Bass-Serre theorem* [4, Theorem I.13] gives the following presentation for $G$.

(a) *Generators.* The elements of all the $G_w$ where $w$ is a vertex of $T$ and the $\gamma_f$ where $f$ is an edge of $X/G$.

(b) *Relations.* The internal relations of each stabilizer $G_w$ together with

$$\gamma_f = 1 \text{ if } f \text{ is an edge of } M,$$

$$\gamma_{\bar{f}} = (\gamma_f)^{-1} \text{ and}$$

$$\gamma_{\bar{f}}\, g_x \gamma_f = (\gamma_{\bar{f}}\, g\, \gamma_f)_z \text{ where } e \text{ is the chosen lift of } f \text{ and } g \in G_e.$$

(If $f$ is an edge of $M$ then $z = t(e)$ and the final relation reduces to $g_x = g_z$ whenever $g \in G_e$).