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TREES, TAIL WAGGING AND GROUP PRESENTATIONS

by M. A. ARMSTRONG

The Bass-Serre theorem gives a presentation for a group of automorphisms of a tree. Like all good theorems it has attracted considerable attention and there are now several proofs available [4], [3], [1]. Our goal is a natural elementary proof which makes maximal use of the geometry of the tree.

1. GRAPHS

A graph X consists of two sets E (directed edges) and V (vertices) and two functions

$$\begin{aligned} E &\rightarrow E, & e &\mapsto \bar{e} \\ E &\rightarrow V \times V, & e &\mapsto (i(e), t(e)) \end{aligned}$$

which satisfy $\bar{\bar{e}} = e$, $\bar{e} \neq e$ and $i(\bar{e}) = t(e)$ for each $e \in E$. The vertices $i(e)$, $t(e)$ are the initial and terminal vertices of the directed edge e , and \bar{e} is the reverse of e . Henceforth we refer to directed edges simply as edges.

A path in X joining vertex u to vertex v is an ordered string of edges $e_1 e_2 \dots e_n$ such that $i(e_1) = u$, $i(e_{k+1}) = t(e_k)$ for $1 \leq k \leq n-1$, and $t(e_n) = v$. If $v = u$ we have a circuit. A path of the form $e\bar{e}$ is a *round trip* and a circuit which does not contain any round trips will be called a loop. If any two distinct vertices may be joined by a path then the graph is connected. A *tree* is a connected graph which does not contain any loops.

Let X be a tree. A path in X is a *geodesic* if it does not contain any round trips. Given distinct vertices u, v of X there is a *unique* geodesic \overrightarrow{uv} which joins u to v .

An action of a group G on a graph X is an action of G on E and on V such that $g\bar{e} = \overline{ge}$, $i(ge) = gi(e)$, $t(ge) = gt(e)$ and $ge \neq \bar{e}$ for each $e \in E$. Because group elements are not allowed to reverse edges we have a

quotient graph X/G . When G acts on X we shall often say that G is a group of automorphisms of X .

We adopt the usual notation whereby G_x denotes the stabilizer of a vertex x . If $g \in G$ happens to fix x we write g_x for the element g thought of as a member of G_x . Of course G_e denotes the stabilizer of the edge e . If x is a vertex of e then G_e is a subgroup of G_x .

Suppose G acts on a tree X . If $g \in G$ fixes the vertices u, v then it must fix the whole geodesic \overrightarrow{uv} , since otherwise the image of \overrightarrow{uv} under g would be a second geodesic from u to v .

2. LIFTING EDGES

Let G be a group of automorphisms of a tree X . Choose a maximal tree M in X/G and lift it [4, Proposition I.14] to a subtree T of X . The vertices of T form a set of representatives for the action of G on the vertices of X . For each pair of edges f, \bar{f} from $X/G - M$ select one, say f , and lift it to an edge e of X which has its initial vertex x in T . Exactly one vertex z of T lies in the same orbit as $t(e)$ and we choose an element γ_f from G that maps z onto $t(e)$. We can now lift \bar{f} to $(\gamma_f)^{-1}\bar{e}$. This has its initial vertex z in T and $\gamma_{\bar{f}} = (\gamma_f)^{-1}$ sends the vertex x of T to its terminal vertex (Figure 1). Finally we extend the correspondence $f \rightarrow \gamma_f$ over the edges of M by setting $\gamma_f = 1$ (the identity element of G) whenever $f \in M$.

The Bass-Serre theorem [4, Theorem I.13] gives the following presentation for G .

- (a) *Generators.* The elements of all the G_w where w is a vertex of T and the γ_f where f is an edge of X/G .
- (b) *Relations.* The internal relations of each stabilizer G_w together with

$$\gamma_f = 1 \text{ if } f \text{ is an edge of } M,$$

$$\gamma_{\bar{f}} = (\gamma_f)^{-1} \text{ and}$$

$$\gamma_{\bar{f}} g_x \gamma_f = (\gamma_{\bar{f}} g \gamma_f)_z \text{ where } e \text{ is the chosen lift of } f \text{ and } g \in G_e.$$

(If f is an edge of M then $z = t(e)$ and the final relation reduces to $g_x = g_z$ whenever $g \in G_e$).

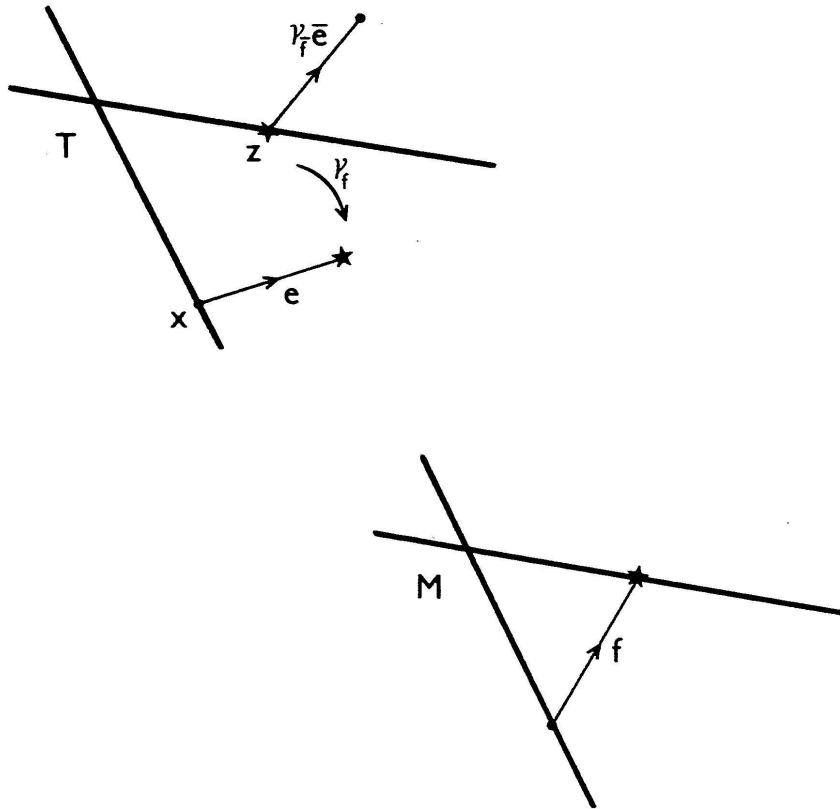


FIGURE 1

3. TAIL WAGGING

With the notation established above let $*G_w$ denote the free product of the stabilizers of the vertices of T , and F the free group generated by symbols λ_f , one for each edge f of X/G . Let R be the normal consequence in $(*G_w)*F$ of the words

$$\begin{aligned} \lambda_f & \quad (f \text{ an edge of } M), \\ \lambda_{\bar{f}} \lambda_f & \quad \text{and} \\ \lambda_{\bar{f}} g_x \lambda_f (\gamma_{\bar{f}} g \gamma_f)_z^{-1} \end{aligned}$$

We shall produce an isomorphism

$$\psi: G \rightarrow [(*G_w)*F]/R.$$

Choose a vertex v of T as base point. If $g \in G$ fixes v set

$$\psi(g) = g_v R$$

where as usual g_v is the element g interpreted as a member of G_v . If g moves v then it sends it outside T because no two vertices of T lie in the same orbit. Let $e_1 e_2 \dots e_n$ be the geodesic which joins v to gv and suppose e_m is the first edge that is *not* in T . The path $e_m e_{m+1} \dots e_n$ will be called the *tail* of $\overrightarrow{v gv}$. Let x_1 be the initial vertex of e_m . Project e_m into X/G to give an edge f_1 . The canonical lift e^1 of f_1 into X has its initial vertex in T , so $i(e^1) = x_1$. Choose an element $a_{x_1} \in G_{x_1}$ which sends e^1 to e_m . Let

$$e_k^1 = (\gamma_{f_1} a_{x_1}^{-1}) e_k$$

for $m+1 \leq k \leq n$, and replace $e_1 e_2 \dots e_n$ by the new path $e_{m+1}^1 e_{m+2}^1 \dots e_n^1$. We call this process *tail wagging*. Our new path begins at

$$z_1 = t(\gamma_{f_1} e^1) = i(e_{m+1}^1)$$

which is a vertex of T and ends at $(\gamma_{f_1} a_{x_1}^{-1} g)v$, see Figure 2. We walk along it to the first point x_2 where it quits T and repeat the above

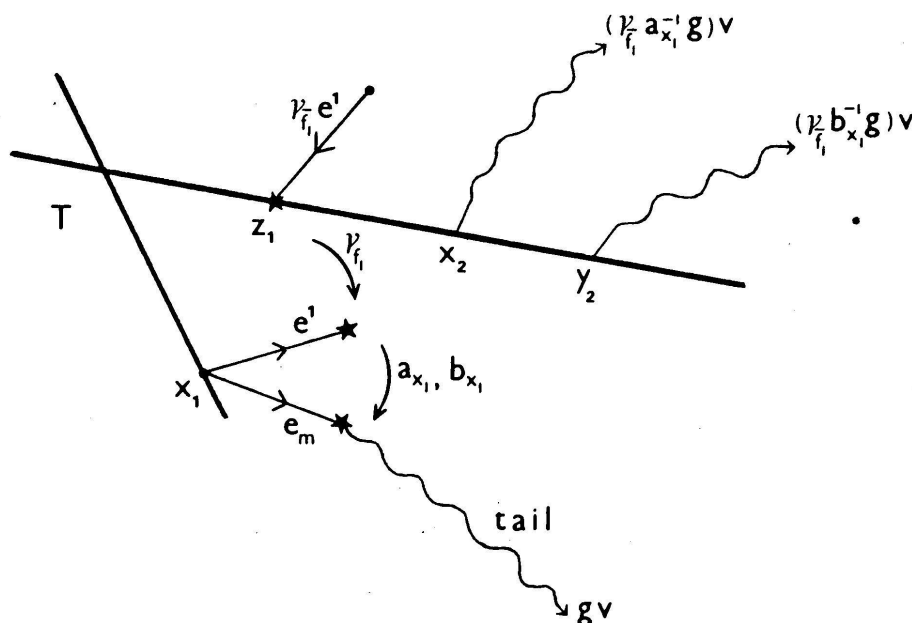


FIGURE 2

procedure. Since we shorten the tail at each step we eventually obtain a path which lies entirely in T and ends at say

$$(\gamma_{\bar{f}_r} a_{x_r}^{-1} \dots \gamma_{\bar{f}_2} a_{x_2}^{-1} \gamma_{\bar{f}_1} a_{x_1}^{-1} g)v.$$

Then $\gamma_{\bar{f}_r} a_{x_r}^{-1} \dots \gamma_{\bar{f}_1} a_{x_1}^{-1} g$ must fix v , say $\gamma_{\bar{f}_r} a_{x_r}^{-1} \dots \gamma_{\bar{f}_1} a_{x_1}^{-1} g = a_v \in G_v$.

We now have

$$g = a_{x_1} \gamma_{f_1} \dots a_{x_r} \gamma_{f_r} a_v$$

and we somewhat optimistically define

$$\psi(g) = a_{x_1} \lambda_{f_1} \dots a_{x_r} \lambda_{f_r} a_v R.$$

4. AN INEFFICIENT CHOICE

Is ψ well defined? The geodesic from v to gv is certainly unique, as is the first point x_1 where it leaves T and its first edge e_m outside T . Both the edge e^1 and the group element γ_{f_1} are now determined by our original construction. The only ambiguity at this stage is the choice of the element $a_{x_1} \in G_{x_1}$ which maps e^1 to e_m . A different choice b_{x_1} will give a path from z_1 to $(\gamma_{\bar{f}_1} b_{x_1}^{-1} g)v$ which leaves T for the first time at say y_2 . The first edge outside T will project to an edge f'_2 of X/G and so on until eventually we have g expressed as

$$g = b_{x_1} \gamma_{f_1} b_{y_2} \gamma_{f'_2} \dots b_{y_s} \gamma_{f'_s} b_v.$$

We must show that $a_{x_1} \lambda_{f_1} a_{x_2} \lambda_{f_2} \dots a_{x_r} \lambda_{f_r} a_v$ and $b_{x_1} \lambda_{f_1} b_{y_2} \lambda_{f'_2} \dots b_{y_s} \lambda_{f'_s} b_v$ determine the same left coset of R in $(*G_w)*F$.

Agree to select a_{x_1} from G_{x_1} so that the tail of the resulting path is as long as possible. Continue in this way selecting $a_{x_2}, a_{x_3} \dots$ so as to maximise the length of the tail at each stage. We shall compare any other set of choices with this rather inefficient selection.

Both a_{x_1} and b_{x_1} map e^1 to e_m , so $c = a_{x_1}^{-1} b_{x_1}$ must fix e^1 . Also, due to our particular selection of a_{x_1} , the geodesic from z_1 to x_2 is left fixed by $\gamma_{\bar{f}_1} c \gamma_{f_1}$. Therefore

$$\begin{aligned}
& b_{x_1} \lambda_{f_1} b_{y_2} \lambda_{f'_2} \dots b_{y_s} \lambda_{f'_s} b_v R \\
&= a_{x_1} \lambda_{f_1} \lambda_{\bar{f}_1} a_{x_1}^{-1} b_{x_1} \lambda_{f_1} b_{y_2} \lambda_{f'_2} \dots b_{y_s} \lambda_{f'_s} b_v R \\
&= a_{x_1} \lambda_{f_1} \lambda_{\bar{f}_1} c_{x_1} \lambda_{f_1} b_{y_2} \lambda_{f'_2} \dots b_{y_s} \lambda_{f'_s} b_v R \\
&= a_{x_1} \lambda_{f_1} (\gamma_{\bar{f}_1} c \gamma_{f_1})_{x_1} b_{y_2} \lambda_{f'_2} \dots b_{y_s} \lambda_{f'_s} b_v R \\
&= a_{x_1} \lambda_{f_1} (\gamma_{\bar{f}_1} c \gamma_{f_1})_{x_2} b_{y_2} \lambda_{f'_2} \dots b_{y_s} \lambda_{f'_s} b_v R \\
&= a_{x_1} \lambda_{f_1} a'_{x_2} b_{y_2} \lambda_{f'_2} \dots b_{y_s} \lambda_{f'_s} b_v R
\end{aligned}$$

where $a'_{x_2} = (\gamma_{\bar{f}_1} c \gamma_{f_1})_{x_2}$. If x_2 happens to equal y_2 then we simplify this further to

$$a_{x_1} \lambda_{f_1} a''_{x_2} \lambda_{f_2} b_{y_3} \lambda_{f'_3} \dots b_{y_s} \lambda_{f'_s} b_v R$$

where a''_{x_2} is the product $a'_{x_2} b_{y_2}$ in G_{x_2} . We now compare a_{x_2} with a'_{x_2} if $x_2 \neq y_2$, noting that $\gamma_{f_2} = 1$ in this case, or with a''_{x_2} if $x_2 = y_2$, and repeat the process. Eventually we obtain

$$b_{x_1} \lambda_{f_1} b_{y_2} \lambda_{f'_2} \dots b_{y_s} \lambda_{f'_s} b_v R = a_{x_1} \lambda_{f_1} a_{x_2} \lambda_{f_2} \dots a_{x_r} \lambda_{f_r} a''_v R.$$

As $g = a_{x_1} \gamma_{f_1} \dots a_{x_r} \gamma_{f_r} a_v = a_{x_1} \gamma_{f_1} \dots a_{x_r} \gamma_{f_r} a''_v$ we see that $a''_v = a_v$. This completes the proof that ψ is well defined.

5. NEAREST FIXED POINTS

To show ψ is a homomorphism we shall verify

$$\psi(hg) = \psi(h)\psi(g)$$

under the assumption that h either leaves some vertex of T fixed or is one of the elements γ_f . This is sufficient because the elements of the G_w (w a vertex of T) together with the γ_f (f an edge of $X/G-M$) form a set of generators for G .

Suppose h fixes the vertex w of T . Walk along the geodesic \overrightarrow{vw} and let x be the first vertex we meet which is left fixed by h . Then \overrightarrow{vx} is contained in T , and \overrightarrow{vx} followed by $h(\overrightarrow{vx})$ is the geodesic from v to hv . This quits T for the first time at x and we see that

$$\psi(h) = h_x R.$$

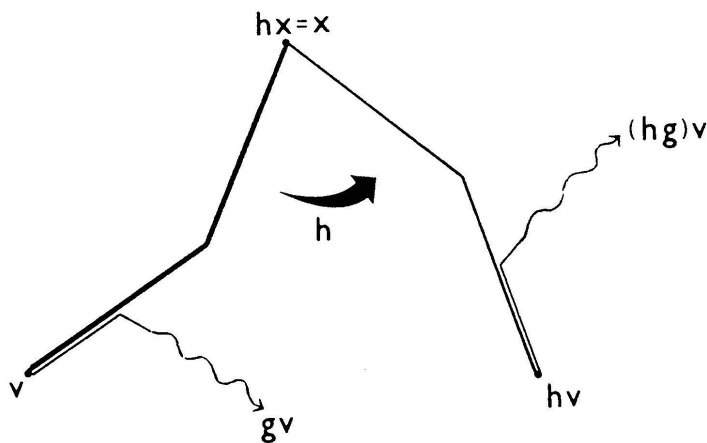


FIGURE 3

Using the geodesic from v to gv we have $\psi(g) = a_{x_1} \lambda_{f_1} \dots a_{x_r} \lambda_{f_r} a_v R$ in the usual way. Therefore

$$\psi(h)\psi(g) = h_x a_{x_1} \lambda_{f_1} \dots a_{x_r} \lambda_{f_r} a_v R.$$

In order to compute $\psi(hg)$ we need the geodesic from v to $(hg)v$. We can construct this as follows, take \overrightarrow{vhv} followed by the image of \overrightarrow{vgv} under h and remove any round trips.

If \overrightarrow{vgv} does not contain all of \overrightarrow{vx} (Figure 3) then $\overrightarrow{v(hg)v}$ leaves T for the first time at x . A tail wag of $\overrightarrow{v(hg)v}$ using h_x^{-1} leads us to a path which has the same tail as \overrightarrow{vgv} , then the process continues as for g . Thus

$$\psi(hg) = h_x a_{x_1} \lambda_{f_1} \dots a_{x_r} \lambda_{f_r} a_v R = \psi(h)\psi(g).$$

Otherwise \overrightarrow{vgv} contains all of \overrightarrow{vx} (Figure 4) and we split the argument into three cases.

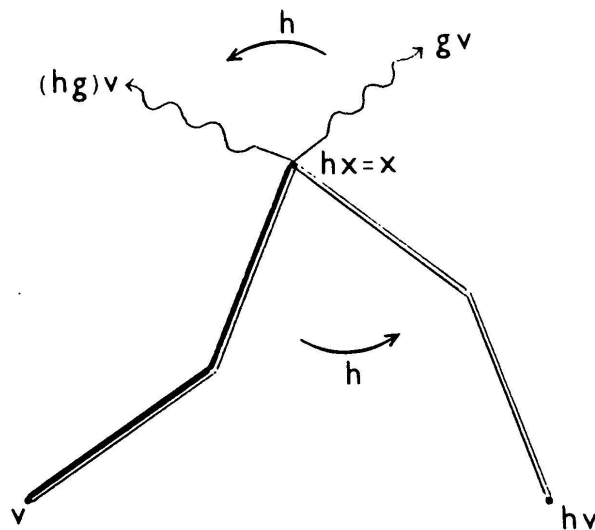


FIGURE 4

produces a path with the same tail as $\overrightarrow{v g v}$. The process then continues as for g and

$$\psi(\gamma_f g) = \lambda_f a_{x_1} \lambda_{f_1} \dots a_{x_r} \lambda_{f_r} a_v R = \psi(\gamma_f) \psi(g).$$

Otherwise $\overrightarrow{v g v}$ contains $\overrightarrow{v y}$. Then $x_1 = z$, $\gamma_{f_1} = \gamma_{\bar{f}}$ and we may as well take $a_{x_1} = 1$. A first tail wag of $\overrightarrow{v g v}$ using γ_f leaves a path with the same tail as $\overrightarrow{v(\gamma_f g)v}$. Thus

$$\begin{aligned} \psi(\gamma_f g) &= a_{x_2} \lambda_{f_2} \dots a_{x_r} \lambda_{f_r} a_v R \\ &= \lambda_f \lambda_{\bar{f}} a_{x_2} \lambda_{f_2} \dots a_{x_r} \lambda_{f_r} a_v R \\ &= \psi(\gamma_f) \psi(g). \end{aligned}$$

This completes the proof that ψ is a homomorphism.

Our construction of ψ ensures that if $\psi(g) = R$ then $g = 1$. So ψ is injective. The cosets $h_w R$ (w a vertex of T and $h(w) = w$) and $\lambda_f R$ (f an edge of X/G) together generate $[(\ast G_w) \ast F]/R$. Now $\psi(h) = h_x R$ where x is the nearest fixed point of h to v . But h fixes all of \overrightarrow{xw} so

$$\psi(h) = h_x R = h_w R.$$

Also

$$\psi(\gamma_f) = \lambda_f R.$$

Therefore the image of ψ is all of $[(\ast G_w) \ast F]/R$ and we have shown that ψ is an isomorphism.

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