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## TREES, TAIL WAGGING AND GROUP PRESENTATIONS

by M. A. ARMSTRONG

The Bass-Serre theorem gives a presentation for a group of automorphisms of a tree. Like all good theorems it has attracted considerable attention and there are now several proofs available [4], [3], [1]. Our goal is a natural elementary proof which makes maximal use of the geometry of the tree.

### 1. GRAPHS

A *graph*  $X$  consists of two sets  $E$  (directed edges) and  $V$  (vertices) and two functions

$$\begin{aligned} E &\rightarrow E, \quad e \mapsto \bar{e} \\ E &\rightarrow V \times V, \quad e \mapsto (i(e), t(e)) \end{aligned}$$

which satisfy  $\bar{\bar{e}} = e$ ,  $\bar{e} \neq e$  and  $i(\bar{e}) = t(e)$  for each  $e \in E$ . The vertices  $i(e)$ ,  $t(e)$  are the initial and terminal vertices of the directed edge  $e$ , and  $\bar{e}$  is the reverse of  $e$ . Henceforth we refer to directed edges simply as edges.

A path in  $X$  joining vertex  $u$  to vertex  $v$  is an ordered string of edges  $e_1 e_2 \dots e_n$  such that  $i(e_1) = u$ ,  $i(e_{k+1}) = t(e_k)$  for  $1 \leq k \leq n-1$ , and  $t(e_n) = v$ . If  $v = u$  we have a circuit. A path of the form  $e\bar{e}$  is a *round trip* and a circuit which does not contain any round trips will be called a loop. If any two distinct vertices may be joined by a path then the graph is connected. A *tree* is a connected graph which does not contain any loops.

Let  $X$  be a tree. A path in  $X$  is a *geodesic* if it does not contain any round trips. Given distinct vertices  $u, v$  of  $X$  there is a *unique* geodesic  $\overrightarrow{uv}$  which joins  $u$  to  $v$ .

An action of a group  $G$  on a graph  $X$  is an action of  $G$  on  $E$  and on  $V$  such that  $g\bar{e} = \bar{ge}$ ,  $i(ge) = gi(e)$ ,  $t(ge) = gt(e)$  and  $ge \neq \bar{e}$  for each  $e \in E$ . Because group elements are not allowed to reverse edges we have a

quotient graph  $X/G$ . When  $G$  acts on  $X$  we shall often say that  $G$  is a *group of automorphisms of  $X$* .

We adopt the usual notation whereby  $G_x$  denotes the stabilizer of a vertex  $x$ . If  $g \in G$  happens to fix  $x$  we write  $g_x$  for the element  $g$  thought of as a member of  $G_x$ . Of course  $G_e$  denotes the stabilizer of the edge  $e$ . If  $x$  is a vertex of  $e$  then  $G_e$  is a subgroup of  $G_x$ .

Suppose  $G$  acts on a tree  $X$ . If  $g \in G$  fixes the vertices  $u, v$  then it must fix the whole geodesic  $\overrightarrow{uv}$ , since otherwise the image of  $\overrightarrow{uv}$  under  $g$  would be a second geodesic from  $u$  to  $v$ .

## 2. LIFTING EDGES

Let  $G$  be a group of automorphisms of a tree  $X$ . Choose a maximal tree  $M$  in  $X/G$  and lift it [4, Proposition I.14] to a subtree  $T$  of  $X$ . The vertices of  $T$  form a set of representatives for the action of  $G$  on the vertices of  $X$ . For each pair of edges  $f, \bar{f}$  from  $X/G - M$  select one, say  $f$ , and lift it to an edge  $e$  of  $X$  which has its initial vertex  $x$  in  $T$ . Exactly one vertex  $z$  of  $T$  lies in the same orbit as  $t(e)$  and we choose an element  $\gamma_f$  from  $G$  that maps  $z$  onto  $t(e)$ . We can now lift  $\bar{f}$  to  $(\gamma_f)^{-1}\bar{e}$ . This has its initial vertex  $z$  in  $T$  and  $\gamma_{\bar{f}} = (\gamma_f)^{-1}$  sends the vertex  $x$  of  $T$  to its terminal vertex (Figure 1). Finally we extend the correspondence  $f \rightarrow \gamma_f$  over the edges of  $M$  by setting  $\gamma_f = 1$  (the identity element of  $G$ ) whenever  $f \in M$ .

The *Bass-Serre theorem* [4, Theorem I.13] gives the following presentation for  $G$ .

- (a) *Generators.* The elements of all the  $G_w$  where  $w$  is a vertex of  $T$  and the  $\gamma_f$  where  $f$  is an edge of  $X/G$ .
- (b) *Relations.* The internal relations of each stabilizer  $G_w$  together with
  - $\gamma_f = 1$  if  $f$  is an edge of  $M$ ,
  - $\gamma_{\bar{f}} = (\gamma_f)^{-1}$  and

$\gamma_{\bar{f}} g_x \gamma_f = (\gamma_{\bar{f}} g \gamma_f)_z$  where  $e$  is the chosen lift of  $f$  and  $g \in G_e$ . (If  $f$  is an edge of  $M$  then  $z = t(e)$  and the final relation reduces to  $g_x = g_z$  whenever  $g \in G_e$ ).

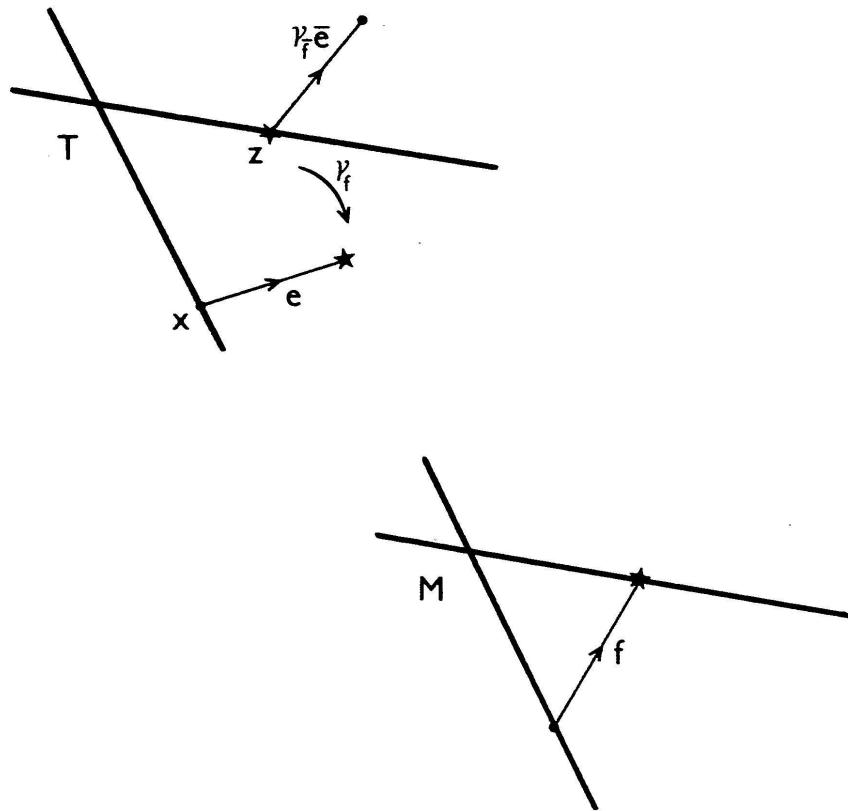


FIGURE 1

## 3. TAIL WAGGING

With the notation established above let  $*G_w$  denote the free product of the stabilizers of the vertices of  $T$ , and  $F$  the free group generated by symbols  $\lambda_f$ , one for each edge  $f$  of  $X/G$ . Let  $R$  be the normal consequence in  $(*G_w)*F$  of the words

$$\begin{aligned} \lambda_f & \quad (f \text{ an edge of } M), \\ \lambda_{\bar{f}} \lambda_f & \quad \text{and} \\ \lambda_{\bar{f}} g_x \lambda_f (\gamma_{\bar{f}} g \gamma_f)^{-1} \end{aligned}$$

We shall produce an isomorphism

$$\psi: G \rightarrow [(*G_w)*F]/R.$$

Choose a vertex  $v$  of  $T$  as base point. If  $g \in G$  fixes  $v$  set

$$\psi(g) = g_v R$$

where as usual  $g_v$  is the element  $g$  interpreted as a member of  $G_v$ . If  $g$  moves  $v$  then it sends it outside  $T$  because no two vertices of  $T$  lie in the same orbit. Let  $e_1 e_2 \dots e_n$  be the geodesic which joins  $v$  to  $gv$  and suppose  $e_m$  is the first edge that is *not* in  $T$ . The path  $e_m e_{m+1} \dots e_n$  will be called the *tail* of  $\overrightarrow{v\,gv}$ . Let  $x_1$  be the initial vertex of  $e_m$ . Project  $e_m$  into  $X/G$  to give an edge  $f_1$ . The canonical lift  $e^1$  of  $f_1$  into  $X$  has its initial vertex in  $T$ , so  $i(e^1) = x_1$ . Choose an element  $a_{x_1} \in G_{x_1}$  which sends  $e^1$  to  $e_m$ . Let

$$e_k^1 = (\gamma_{\bar{f}_1} a_{x_1}^{-1}) e_k$$

for  $m+1 \leq k \leq n$ , and replace  $e_1 e_2 \dots e_n$  by the new path  $e_{m+1}^1 e_{m+2}^1 \dots e_n^1$ . We call this process *tail wagging*. Our new path begins at

$$z_1 = t(\gamma_{\bar{f}_1} e^1) = i(e^1_{m+1})$$

which is a vertex of  $T$  and ends at  $(\gamma_{\bar{f}_1} a_{x_1}^{-1} g)v$ , see Figure 2. We walk along it to the first point  $x_2$  where it quits  $T$  and repeat the above

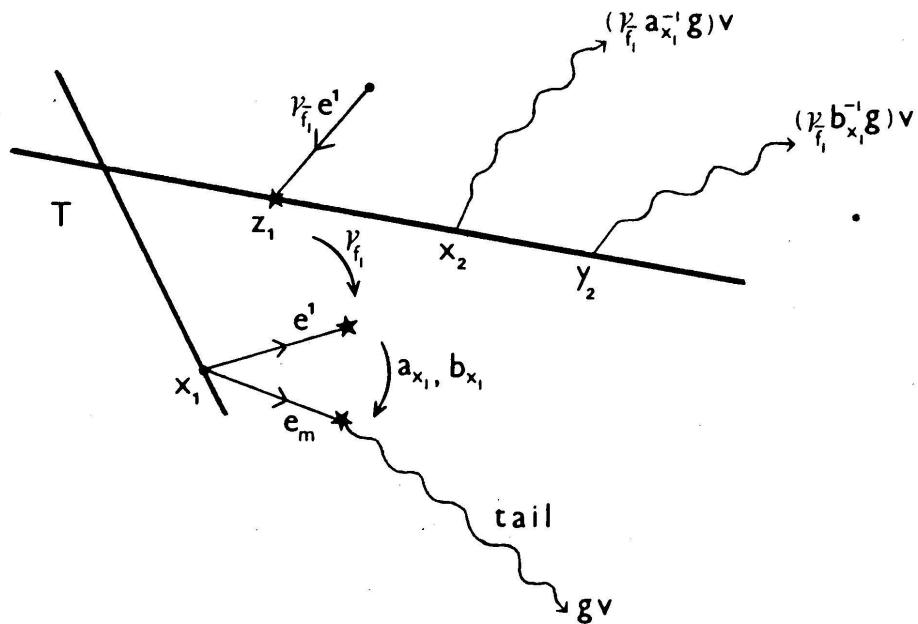


FIGURE 2

procedure. Since we shorten the tail at each step we eventually obtain a path which lies entirely in  $T$  and ends at say

$$(\gamma_{\bar{f}_r} a_{x_r}^{-1} \dots \gamma_{\bar{f}_2} a_{x_2}^{-1} \gamma_{\bar{f}_1} a_{x_1}^{-1} g)v.$$

Then  $\gamma_{\bar{f}_r} a_{x_r}^{-1} \dots \gamma_{\bar{f}_1} a_{x_1}^{-1} g$  must fix  $v$ , say  $\gamma_{\bar{f}_r} a_{x_r}^{-1} \dots \gamma_{\bar{f}_1} a_{x_1}^{-1} g = a_v \in G_v$ . We now have

$$g = a_{x_1} \gamma_{f_1} \dots a_{x_r} \gamma_{f_r} a_v$$

and we somewhat optimistically define

$$\psi(g) = a_{x_1} \lambda_{f_1} \dots a_{x_r} \lambda_{f_r} a_v R.$$

#### 4. AN INEFFICIENT CHOICE

Is  $\psi$  well defined? The geodesic from  $v$  to  $gv$  is certainly unique, as is the first point  $x_1$  where it leaves  $T$  and its first edge  $e_m$  outside  $T$ . Both the edge  $e^1$  and the group element  $\gamma_{f_1}$  are now determined by our original construction. The only ambiguity at this stage is the choice of the element  $a_{x_1} \in G_{x_1}$  which maps  $e^1$  to  $e_m$ . A different choice  $b_{x_1}$  will give a path from  $z_1$  to  $(\gamma_{\bar{f}_1} b_{x_1}^{-1} g)v$  which leaves  $T$  for the first time at say  $y_2$ . The first edge outside  $T$  will project to an edge  $f'_2$  of  $X/G$  and so on until eventually we have  $g$  expressed as

$$g = b_{x_1} \gamma_{f_1} b_{y_2} \gamma_{f'_2} \dots b_{y_s} \gamma_{f'_s} b_v.$$

We must show that  $a_{x_1} \lambda_{f_1} a_{x_2} \lambda_{f_2} \dots a_{x_r} \lambda_{f_r} a_v$  and  $b_{x_1} \lambda_{f_1} b_{y_2} \lambda_{f'_2} \dots b_{y_s} \lambda_{f'_s} b_v$  determine the same left coset of  $R$  in  $(*G_w)*F$ .

Agree to select  $a_{x_1}$  from  $G_{x_1}$  so that the tail of the resulting path is as long as possible. Continue in this way selecting  $a_{x_2}, a_{x_3} \dots$  so as to maximise the length of the tail at each stage. We shall compare any other set of choices with this rather inefficient selection.

Both  $a_{x_1}$  and  $b_{x_1}$  map  $e^1$  to  $e_m$ , so  $c = a_{x_1}^{-1} b_{x_1}$  must fix  $e^1$ . Also, due to our particular selection of  $a_{x_1}$ , the geodesic from  $z_1$  to  $x_2$  is left fixed by  $\gamma_{\bar{f}_1} c \gamma_{f_1}$ . Therefore

$$\begin{aligned}
& b_{x_1} \lambda_{f_1} b_{y_2} \lambda_{f'_2} \dots b_{y_s} \lambda_{f'_s} b_v R \\
&= a_{x_1} \lambda_{f_1} \lambda_{\bar{f}_1} a_{x_1}^{-1} b_{x_1} \lambda_{f_1} b_{y_2} \lambda_{f'_2} \dots b_{y_s} \lambda_{f'_s} b_v R \\
&= a_{x_1} \lambda_{f_1} \lambda_{\bar{f}_1} c_{x_1} \lambda_{f_1} b_{y_2} \lambda_{f'_2} \dots b_{y_s} \lambda_{f'_s} b_v R \\
&= a_{x_1} \lambda_{f_1} (\gamma_{\bar{f}_1} c \gamma_{f_1})_{z_1} b_{y_2} \lambda_{f'_2} \dots b_{y_s} \lambda_{f'_s} b_v R \\
&= a_{x_1} \lambda_{f_1} (\gamma_{\bar{f}_1} c \gamma_{f_1})_{x_2} b_{y_2} \lambda_{f'_2} \dots b_{y_s} \lambda_{f'_s} b_v R \\
&= a_{x_1} \lambda_{f_1} a'_{x_2} b_{y_2} \lambda_{f'_2} \dots b_{y_s} \lambda_{f'_s} b_v R
\end{aligned}$$

where  $a'_{x_2} = (\gamma_{\bar{f}_1} c \gamma_{f_1})_{x_2}$ . If  $x_2$  happens to equal  $y_2$  then we simplify this further to

$$a_{x_1} \lambda_{f_1} a''_{x_2} \lambda_{f_2} b_{y_3} \lambda_{f'_3} \dots b_{y_s} \lambda_{f'_s} b_v R$$

where  $a''_{x_2}$  is the product  $a'_{x_2} b_{y_2}$  in  $G_{x_2}$ . We now compare  $a_{x_2}$  with  $a'_{x_2}$  if  $x_2 \neq y_2$ , noting that  $\gamma_{f_2} = 1$  in this case, or with  $a''_{x_2}$  if  $x_2 = y_2$ , and repeat the process. Eventually we obtain

$$b_{x_1} \lambda_{f_1} b_{y_2} \lambda_{f'_2} \dots b_{y_s} \lambda_{f'_s} b_v R = a_{x_1} \lambda_{f_1} a_{x_2} \lambda_{f_2} \dots a_{x_r} \lambda_{f_r} a''_v R.$$

As  $g = a_{x_1} \gamma_{f_1} \dots a_{x_r} \gamma_{f_r} a_v = a_{x_1} \gamma_{f_1} \dots a_{x_r} \gamma_{f_r} a''_v$  we see that  $a''_v = a_v$ . This completes the proof that  $\psi$  is well defined.

## 5. NEAREST FIXED POINTS

To show  $\psi$  is a homomorphism we shall verify

$$\psi(hg) = \psi(h)\psi(g)$$

under the assumption that  $h$  either leaves some vertex of  $T$  fixed or is one of the elements  $\gamma_f$ . This is sufficient because the elements of the  $G_w$  ( $w$  a vertex of  $T$ ) together with the  $\gamma_f$  ( $f$  an edge of  $X/G - M$ ) form a set of generators for  $G$ .

Suppose  $h$  fixes the vertex  $w$  of  $T$ . Walk along the geodesic  $\overrightarrow{vw}$  and let  $x$  be the first vertex we meet which is left fixed by  $h$ . Then  $\overrightarrow{vx}$  is contained in  $T$ , and  $\overrightarrow{vx}$  followed by  $h(\overrightarrow{vx})$  is the geodesic from  $v$  to  $hv$ . This quits  $T$  for the first time at  $x$  and we see that

$$\psi(h) = h_x R.$$

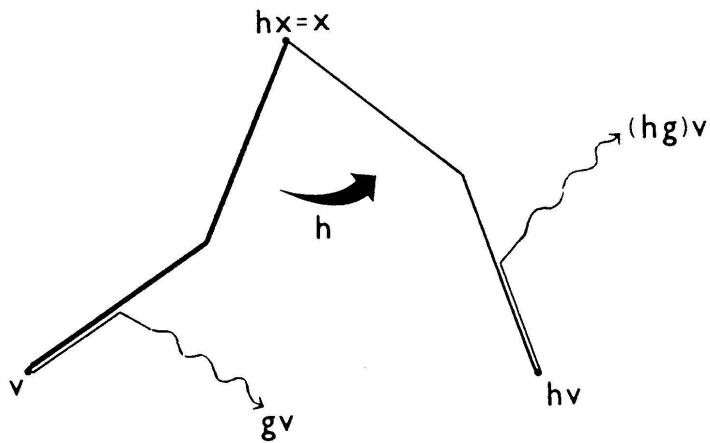


FIGURE 3

Using the geodesic from  $v$  to  $gv$  we have  $\psi(g) = a_{x_1} \lambda_{f_1} \dots a_{x_r} \lambda_{f_r} a_v R$  in the usual way. Therefore

$$\psi(h)\psi(g) = h_x a_{x_1} \lambda_{f_1} \dots a_{x_r} \lambda_{f_r} a_v R.$$

In order to compute  $\psi(hg)$  we need the geodesic from  $v$  to  $(hg)v$ . We can construct this as follows, take  $\overrightarrow{v hv}$  followed by the image of  $\overrightarrow{vgv}$  under  $h$  and remove any round trips.

If  $\overrightarrow{vgv}$  does not contain all of  $\overrightarrow{vx}$  (Figure 3) then  $\overrightarrow{v(hg)v}$  leaves  $T$  for the first time at  $x$ . A tail wag of  $\overrightarrow{v(hg)v}$  using  $h_x^{-1}$  leads us to a path which has the same tail as  $\overrightarrow{vgv}$ , then the process continues as for  $g$ . Thus

$$\psi(hg) = h_x a_{x_1} \lambda_{f_1} \dots a_{x_r} \lambda_{f_r} a_v R = \psi(h)\psi(g).$$

Otherwise  $\overrightarrow{vgv}$  contains all of  $\overrightarrow{vx}$  (Figure 4) and we split the argument into three cases.

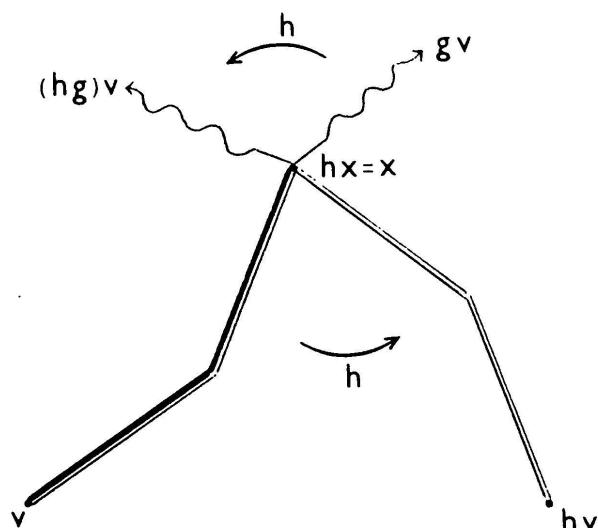


FIGURE 4

(a)  $\overrightarrow{vgv}$  stays in  $T$  for at least one more edge after  $x$ . Then  $\overrightarrow{v(hg)v}$  must leave  $T$  at  $x$ . As above, a first choice of  $h_x^{-1}$  leads to a path with the same tail as  $\overrightarrow{vgv}$ .

(b)  $\overrightarrow{vgv}$  and  $\overrightarrow{v(hg)v}$  both leave  $T$  at  $x$ . Then  $x_1 = x$  and we write  $a_x$  instead of  $a_{x_1}$ . A first tail wag of  $\overrightarrow{v(hg)v}$  using  $\gamma_{\bar{f}_1}(h_x a_x)^{-1}$  produces the same path as a first tail wag of  $\overrightarrow{vgv}$  using  $\gamma_{\bar{f}_1} a_x^{-1}$ . Thus

$$\psi(hg) = h_x a_x \lambda_{f_1} a_{x_2} \lambda_{f_2} \dots a_{x_r} \lambda_{f_r} a_v R = \psi(h)\psi(g).$$

(c)  $\overrightarrow{vgv}$  leaves  $T$  at  $x$ , but  $\overrightarrow{v(hg)v}$  stays in  $T$  for at least one more edge after  $x$ . Then  $x_1 = x$ ,  $\gamma_{f_1} = 1$  and we may as well equate  $a_{x_1}$  with  $h_x^{-1}$ . A first tail wag of  $\overrightarrow{vgv}$  using  $h_x$  gives a path with the same tail as  $\overrightarrow{v(hg)v}$ . Thus

$$\begin{aligned}\psi(hg) &= a_{x_2} \lambda_{f_2} \dots a_{x_r} \lambda_{f_r} a_v R \\ &= h_x h_x^{-1} a_{x_2} \lambda_{f_2} \dots a_{x_r} \lambda_{f_r} a_v R \\ &= \psi(h)\psi(g).\end{aligned}$$

Suppose finally that  $h = \gamma_f$  for some edge  $f$  of  $X/G - M$ . As usual  $e$  is the chosen lift of  $f$  into  $X$  with  $x = i(e) \in T$  and  $z = t(\gamma_{\bar{f}} e)$ . Let  $y = i(\gamma_{\bar{f}} e)$ . The geodesic from  $v$  to  $\gamma_f v$  is made up of  $\overrightarrow{vx}$  followed by  $e$  followed by  $\overrightarrow{\gamma_f(zv)}$ . This leaves  $T$  for the first time at  $x$  and a single tail wag using  $\gamma_{\bar{f}}$  produces  $\overrightarrow{zv}$ . Therefore

$$\psi(\gamma_f) = \lambda_f R.$$

To obtain the geodesic from  $v$  to  $(\gamma_f g)v$  we follow  $\overrightarrow{v\gamma_f v}$  by  $\gamma_f \overrightarrow{(vgv)}$  and then remove any round trips (Figure 5). If  $\overrightarrow{vgv}$  does not contain  $\overrightarrow{vy}$ , then  $\overrightarrow{v(\gamma_f g)v}$  leaves  $T$  for the first time at  $x$  and a single tail wag using  $\gamma_{\bar{f}}$

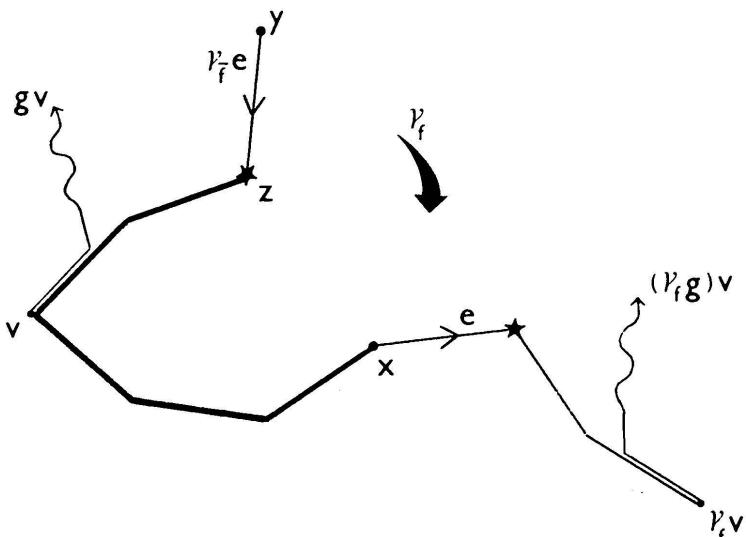


FIGURE 5

produces a path with the same tail as  $\overrightarrow{v gv}$ . The process then continues as for  $g$  and

$$\psi(\gamma_f g) = \lambda_f a_{x_1} \lambda_{f_1} \dots a_{x_r} \lambda_{f_r} a_v R = \psi(\gamma_f) \psi(g).$$

Otherwise  $\overrightarrow{v gv}$  contains  $\overrightarrow{vy}$ . Then  $x_1 = z$ ,  $\gamma_{f_1} = \gamma_{\bar{f}}$  and we may as well take  $a_{x_1} = 1$ . A first tail wag of  $\overrightarrow{v gv}$  using  $\gamma_f$  leaves a path with the same tail as  $\overrightarrow{v(\gamma_f g)v}$ . Thus

$$\begin{aligned} \psi(\gamma_f g) &= a_{x_2} \lambda_{f_2} \dots a_{x_r} \lambda_{f_r} a_v R \\ &= \lambda_f \lambda_{\bar{f}} a_{x_2} \lambda_{f_2} \dots a_{x_r} \lambda_{f_r} a_v R \\ &= \psi(\gamma_f) \psi(g). \end{aligned}$$

This completes the proof that  $\psi$  is a homomorphism.

Our construction of  $\psi$  ensures that if  $\psi(g) = R$  then  $g = 1$ . So  $\psi$  is injective. The cosets  $h_w R$  ( $w$  a vertex of  $T$  and  $h(w) = w$ ) and  $\lambda_f R$  ( $f$  an edge of  $X/G$ ) together generate  $[(*G_w)*F]/R$ . Now  $\psi(h) = h_x R$  where  $x$  is the nearest fixed point of  $h$  to  $v$ . But  $h$  fixes all of  $\overrightarrow{xw}$  so

$$\psi(h) = h_x R = h_w R.$$

Also

$$\psi(\gamma_f) = \lambda_f R.$$

Therefore the image of  $\psi$  is all of  $[(*G_w)*F]/R$  and we have shown that  $\psi$  is an isomorphism.

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