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# TREES, TAIL WAGGING AND GROUP PRESENTATIONS

# by M. A. Armstrong

The Bass-Serre theorem gives a presentation for a group of automorphisms of a tree. Like all good theorems it has attracted considerable attention and there are now several proofs available [4], [3], [1]. Our goal is a natural elementary proof which makes maximal use of the geometry of the tree.

## 1. Graphs

A graph X consists of two sets E (directed edges) and V (vertices) and two functions

$$E \to E, \quad e \mapsto \overline{e}$$
  
 $E \to V \times V, \quad e \mapsto (i(e), t(e))$ 

which satisfy  $\overline{e} = e$ ,  $\overline{e} \neq e$  and  $i(\overline{e}) = t(e)$  for each  $e \in E$ . The vertices i(e), t(e) are the initial and terminal vertices of the directed edge e, and  $\overline{e}$  is the reverse of e. Henceforth we refer to directed edges simply as edges.

A path in X joining vertex u to vertex v is an ordered string of edges  $e_1e_2 \dots e_n$  such that  $i(e_1) = u$ ,  $i(e_{k+1}) = t(e_k)$  for  $1 \le k \le n-1$ , and  $t(e_n) = v$ . If v = u we have a circuit. A path of the form  $e\bar{e}$  is a round trip and a circuit which does not contain any round trips will be called a loop. If any two distinct vertices may be joined by a path then the graph is connected. A tree is a connected graph which does not contain any loops.

Let X be a tree. A path in X is a geodesic if it does not contain any round trips. Given distinct vertices u, v of X there is a unique geodesic  $\overrightarrow{uv}$  which joins u to v.

An action of a group G on a graph X is an action of G on E and on V such that  $g\overline{e} = \overline{ge}$ , i(ge) = gi(e), t(ge) = gt(e) and  $ge \neq \overline{e}$  for each  $e \in E$ . Because group elements are not allowed to reverse edges we have a quotient graph X/G. When G acts on X we shall often say that G is a group of automorphisms of X.

We adopt the usual notation whereby  $G_x$  denotes the stabilizer of a vertex x. If  $g \in G$  happens to fix x we write  $g_x$  for the element g thought of as a member of  $G_x$ . Of course  $G_e$  denotes the stabilizer of the edge e. If x is a vertex of e then  $G_e$  is a subgroup of  $G_x$ .

Suppose G acts on a tree X. If  $g \in G$  fixes the vertices u, v then it must fix the whole geodesic  $\overrightarrow{uv}$ , since otherwise the image of  $\overrightarrow{uv}$  under g would be a second geodesic from u to v.

# 2. LIFTING EDGES

Let G be a group of automorphisms of a tree X. Choose a maximal tree M in X/G and lift it [4, Proposition I.14] to a subtree T of X. The vertices of T form a set of representatives for the action of G on the vertices of X. For each pair of edges  $f, \bar{f}$  from X/G - M select one, say f, and lift it to an edge e of X which has its initial vertex x in T. Exactly one vertex z of T lies in the same orbit as t(e) and we choose an element  $\gamma_f$  from G that maps z onto t(e). We can now lift  $\bar{f}$ to  $(\gamma_f)^{-1}\bar{e}$ . This has its initial vertex z in T and  $\gamma_{\bar{f}} = (\gamma_f)^{-1}$  sends the vertex x of T to its terminal vertex (Figure 1). Finally we extend the correspondence  $f \to \gamma_f$  over the edges of M by setting  $\gamma_f = 1$  (the identity element of G) whenever  $f \in M$ .

The Bass-Serre theorem [4, Theorem I.13] gives the following presentation for G.

- (a) Generators. The elements of all the  $G_w$  where w is a vertex of T and the  $\gamma_f$  where f is an edge of X/G.
- (b) Relations. The internal relations of each stabilizer  $G_w$  together with  $\gamma_f = 1$  if f is an edge of M,  $\gamma_{\bar{f}} = (\gamma_f)^{-1}$  and

 $\gamma_{\bar{f}} g_x \gamma_f = (\gamma_{\bar{f}} g \gamma_f)_z$  where *e* is the chosen lift of *f* and  $g \in G_e$ . (If *f* is an edge of *M* then z = t(e) and the final relation reduces to  $g_x = g_z$  whenever  $g \in G_e$ ).







# 3. TAIL WAGGING

With the notation established above let  $*G_w$  denote the free product of the stablizers of the vertices of T, and F the free group generated by symbols  $\lambda_f$ , one for each edge f of X/G. Let R be the normal consequence in  $(*G_w)*F$  of the words

 $\begin{array}{ll}\lambda_f & (f \text{ an edge of } M),\\ & \lambda_{\bar{f}} \lambda_f & \text{and}\\ & \lambda_{\bar{f}} g_x \lambda_f (\gamma_{\bar{f}} g \gamma_f)_z^{-1} \end{array}$ 

We shall produce an isomorphism

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$$\psi\colon G\to [(\ast G_w)\ast F]/R$$

Choose a vertex v of T as base point. If  $g \in G$  fixes v set

$$\psi(g) = g_v R$$

where as usual  $g_v$  is the element g interpreted as a member of  $G_v$ . If g moves v then it sends it outside T because no two vertices of T lie in the same orbit. Let  $e_1 e_2 \dots e_n$  be the geodesic which joins v to gv and suppose  $e_m$  is the first edge that is not in T. The path  $e_m e_{m+1} \dots e_n$  will be called the *tail* of  $\overrightarrow{v gv}$ . Let  $x_1$  be the initial vertex of  $e_m$ . Project  $e_m$  into X/G to give an edge  $f_1$ . The canonical lift  $e^1$  of  $f_1$  into X has its initial vertex in T, so  $i(e^1) = x_1$ . Choose an element  $a_{x_1} \in G_{x_1}$  which sends  $e^1$  to  $e_m$ . Let

$$e_k^1 = (\gamma_{\bar{f}_1} a_{x_1}^{-1})e_k$$

for  $m+1 \leq k \leq n$ , and replace  $e_1 e_2 \dots e_n$  by the new path  $e_{m+1}^1 e_{m+2}^1 \dots e_n^1$ . We call this process *tail wagging*. Our new path begins at

$$z_1 = t(\gamma_{\bar{f}_1} e^1) = i(e_{m+1}^1)$$

which is a vertex of T and ends at  $(\gamma_{\bar{f}_1} a_{x_1}^{-1} g)v$ , see Figure 2. We walk along it to the first point  $x_2$  where it quits T and repeat the above



FIGURE 2

procedure. Since we shorten the tail at each step we eventually obtain a path which lies entirely in T and ends at say

$$(\gamma_{\bar{f}_r} a_{x_r}^{-1} \dots \gamma_{\bar{f}_2} a_{x_2}^{-1} \gamma_{\bar{f}_1} a_{x_1}^{-1} g) v$$
.

Then  $\gamma_{\bar{f}_r} a_{x_r}^{-1} \dots \gamma_{\bar{f}_1} a_{x_1}^{-1} g$  must fix v, say  $\gamma_{\bar{f}_r} a_{x_r}^{-1} \dots \gamma_{\bar{f}_1} a_{x_1}^{-1} g = a_v \in G_v$ . We now have

$$g = a_{x_1} \gamma_{f_1} \dots a_{x_r} \gamma_{f_r} a_v$$

and we somewhat optimistically define

$$\psi(g) = a_{x_1} \lambda_{f_1} \dots a_{x_r} \lambda_{f_r} a_v R .$$

### 4. AN INEFFICIENT CHOICE

Is  $\psi$  well defined? The geodesic from v to gv is certainly unique, as is the first point  $x_1$  where it leaves T and its first edge  $e_m$  outside T. Both the edge  $e^1$  and the group element  $\gamma_{f_1}$  are now determined by our original construction. The only ambiguity at this stage is the choice of the element  $a_{x_1} \in G_{x_1}$  which maps  $e^1$  to  $e_m$ . A different choice  $b_{x_1}$  will give a path from  $z_1$  to  $(\gamma_{f_1} b_{x_1}^{-1} g)v$  which leaves T for the first time at say  $y_2$ . The first edge outside T will project to an edge  $f'_2$  of X/G and so on until eventually we have g expressed as

$$g = b_{x_1} \gamma_{f_1} b_{y_2} \gamma_{f'_2} \dots b_{y_s} \gamma_{f'_s} b_v .$$

We must show that  $a_{x_1} \lambda_{f_1} a_{x_2} \lambda_{f_2} \dots a_{x_r} \lambda_{f_r} a_v$  and  $b_{x_1} \lambda_{f_1} b_{y_2} \lambda_{f_2} \dots b_{y_s} \lambda_{f_s} b_v$ determine the same left coset of R in  $(*G_w)*F$ .

Agree to select  $a_{x_1}$  from  $G_{x_1}$  so that the tail of the resulting path is as *long* as possible. Continue in this way selecting  $a_{x_2}$ ,  $a_{x_3}$ ... so as to maximise the length of the tail at each stage. We shall compare any other set of choices with this rather inefficient selection.

Both  $a_{x_1}$  and  $b_{x_1}$  map  $e^1$  to  $e_m$ , so  $c = a_{x_1}^{-1} b_{x_1}$  must fix  $e^1$ . Also, due to our particular selection of  $a_{x_1}$ , the geodesic from  $z_1$  to  $x_2$  is left fixed by  $\gamma_{\bar{f}_1} c \gamma_{f_1}$ . Therefore

$$b_{x_{1}} \lambda_{f_{1}} b_{y_{2}} \lambda_{f_{2}} \dots b_{y_{s}} \lambda_{f_{s}} b_{v} R$$

$$= a_{x_{1}} \lambda_{f_{1}} \lambda_{\bar{f}_{1}} a_{x_{1}}^{-1} b_{x_{1}} \lambda_{f_{1}} b_{y_{2}} \lambda_{f_{2}} \dots b_{y_{s}} \lambda_{f_{s}} b_{v} R$$

$$= a_{x_{1}} \lambda_{f_{1}} \lambda_{\bar{f}_{1}} c_{x_{1}} \lambda_{f_{1}} b_{y_{2}} \lambda_{f_{2}} \dots b_{y_{s}} \lambda_{f_{s}} b_{v} R$$

$$= a_{x_{1}} \lambda_{f_{1}} (\gamma_{\bar{f}_{1}} c\gamma_{f_{1}})_{z_{1}} b_{y_{2}} \lambda_{f_{2}} \dots b_{y_{s}} \lambda_{f_{s}} b_{v} R$$

$$= a_{x_{1}} \lambda_{f_{1}} (\gamma_{\bar{f}_{1}} c\gamma_{f_{1}})_{x_{2}} b_{y_{2}} \lambda_{f_{2}} \dots b_{y_{s}} \lambda_{f_{s}} b_{v} R$$

$$= a_{x_{1}} \lambda_{f_{1}} (\gamma_{\bar{f}_{1}} c\gamma_{f_{1}})_{x_{2}} b_{y_{2}} \lambda_{f_{2}} \dots b_{y_{s}} \lambda_{f_{s}} b_{v} R$$

where  $a'_{x_2} = (\gamma_{\bar{f}_1} c \gamma_{f_1})_{x_2}$ . If  $x_2$  happens to equal  $y_2$  then we simplify this further to

$$a_{x_1} \lambda_{f_1} a_{x_2}'' \lambda_{f_2} b_{y_3} \lambda_{f_3}' \dots b_{y_s} \lambda_{f_s}' b_v R$$

where  $a''_{x_2}$  is the product  $a'_{x_2} b_{y_2}$  in  $G_{x_2}$ . We now compare  $a_{x_2}$  with  $a'_{x_2}$  if  $x_2 \neq y_2$ , noting that  $\gamma_{f_2} = 1$  in this case, or with  $a''_{x_2}$  if  $x_2 = y_2$ , and repeat the process. Eventually we obtain

 $b_{x_1}\,\lambda_{f_1}\,b_{y_2}\,\lambda_{f_2}'\,\ldots\,b_{y_s}\,\lambda_{f_s}'\,b_vR\ =\ a_{x_1}\,\lambda_{f_1}\,a_{x_2}\,\lambda_{f_2}\,\ldots\,a_{x_r}\,\lambda_{f_r}\,a_v''\,R\ .$ 

As  $g = a_{x_1} \gamma_{f_1} \dots a_{x_r} \gamma_{f_r} a_v = a_{x_1} \gamma_{f_1} \dots a_{x_r} \gamma_{f_r} a_v''$  we see that  $a_v'' = a_v$ . This completes the proof that  $\psi$  is well defined.

### 5. Nearest fixed points

To show  $\psi$  is a homomorphism we shall verify

$$\psi(hg) = \psi(h)\psi(g)$$

under the assumption that h either leaves some vertex of T fixed or is one of the elements  $\gamma_f$ . This is sufficient because the elements of the  $G_w$  (w a vertex of T) together with the  $\gamma_f$  (f an edge of X/G-M) form a set of generators for G.

Suppose h fixes the vertex w of T. Walk along the geodesic  $\overrightarrow{vw}$  and let x be the first vertex we meet which is left fixed by h. Then  $\overrightarrow{vx}$  is contained in T, and  $\overrightarrow{vx}$  followed by  $h(\overrightarrow{xv})$  is the geodesic from v to hv. This quits T for the first time at x and we see that



FIGURE 3

Using the geodesic from v to gv we have  $\psi(g) = a_{x_1} \lambda_{f_1} \dots a_{x_r} \lambda_{f_r} a_v R$  in the usual way. Therefore

$$\psi(h)\psi(g) = h_x a_{x_1} \lambda_{f_1} \dots a_{x_r} \lambda_{f_r} a_v R.$$

In order to compute  $\psi(hg)$  we need the geodesic from v to (hg)v. We can construct this as follows, take  $\overrightarrow{v h v}$  followed by the image of  $\overrightarrow{v g v}$  under h and remove any round trips.

If  $\overrightarrow{v gv}$  does not contain all of  $\overrightarrow{vx}$  (Figure 3) then  $\overrightarrow{v(hg)v}$  leaves T for the first time at x. A tail wag of  $\overrightarrow{v(hg)v}$  using  $h_x^{-1}$  leads us to a path which has the same tail as  $\overrightarrow{v gv}$ , then the process continues as for g. Thus

$$\psi(hg) = h_x a_{x_1} \lambda_{f_1} \dots a_{x_r} \lambda_{f_r} a_v R = \psi(h) \psi(g) .$$

Otherwise  $\overrightarrow{v g v}$  contains all of  $\overrightarrow{vx}$  (Figure 4) and we split the argument into three cases.



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- (a)  $\overrightarrow{v gv}$  stays in T for at least one more edge after x. Then  $\overrightarrow{v(hg)v}$  must leave T at x. As above, a first choice of  $h_x^{-1}$  leads to a path with the same tail as  $\overrightarrow{v gv}$ .
- (b)  $\overrightarrow{v gv}$  and  $\overrightarrow{v(hg)v}$  both leave T at x. Then  $x_1 = x$  and we write  $a_x$  instead of  $a_{x_1}$ . A first tail wag of  $\overrightarrow{v(hg)v}$  using  $\gamma_{\overline{f_1}}(h_x a_x)^{-1}$  produces the same path as a first tail wag of  $\overrightarrow{v gv}$  using  $\gamma_{\overline{f_1}} a_x^{-1}$ . Thus

$$\Psi(hg) = h_x a_x \lambda_{f_1} a_{x_2} \lambda_{f_2} \dots a_{x_r} \lambda_{f_r} a_v R = \Psi(h) \Psi(g) .$$

(c)  $\overrightarrow{v gv}$  leaves T at x, but  $\overrightarrow{v(hg)v}$  stays in T for at least one more edge after x. Then  $x_1 = x$ ,  $\gamma_{f_1} = 1$  and we may as well equate  $a_{x_1}$  with  $h_x^{-1}$ . A first tail wag of  $\overrightarrow{v gv}$  using  $h_x$  gives a path with the same tail as  $\overrightarrow{v(hg)v}$ . Thus

$$\begin{split} \psi(hg) &= a_{x_2} \lambda_{f_2} \dots a_{x_r} \lambda_{f_r} a_v R \\ &= h_x h_x^{-1} a_{x_2} \lambda_{f_2} \dots a_{x_r} \lambda_{f_r} a_v R \\ &= \psi(h) \psi(g) \,. \end{split}$$

Suppose finally that  $h = \gamma_f$  for some edge f of X/G-M. As usual e is the chosen lift of f into X with  $x = i(e) \in T$  and  $z = t(\gamma_{\bar{f}} e)$ . Let  $y = i(\gamma_{\bar{f}} e)$ . The geodesic from v to  $\gamma_f v$  is made up of  $\overrightarrow{vx}$  followed by e followed by  $\gamma_f(\overrightarrow{zv})$ . This leaves T for the first time at x and a single tail wag using  $\gamma_{\bar{f}}$  produces  $\overrightarrow{zv}$ . Therefore

$$\psi(\gamma_f) = \lambda_f R \, .$$

To obtain the geodesic from v to  $(\gamma_f g)v$  we follow  $\overrightarrow{v\gamma_f v}$  by  $\gamma_f(\overrightarrow{v gv})$ and then remove any round trips (Figure 5). If  $\overrightarrow{v gv}$  does not contain  $\overrightarrow{vy}$ , then  $\overrightarrow{v(\gamma_f g)v}$  leaves T for the first time at x and a single tail wag using  $\gamma_f$ 



produces a path with the same tail as  $\overrightarrow{v gv}$ . The process then continues as for g and

 $\psi(\gamma_f g) = \lambda_f a_{x_1} \lambda_{f_1} \dots a_{x_r} \lambda_{f_r} a_v R = \psi(\gamma_f) \psi(g) .$ 

Otherwise  $\overrightarrow{v gv}$  contains  $\overrightarrow{vy}$ . Then  $x_1 = z$ ,  $\gamma_{f_1} = \gamma_{\overline{f}}$  and we may as well take  $a_{x_1} = 1$ . A first tail wag of  $\overrightarrow{v gv}$  using  $\gamma_f$  leaves a path with the same tail as  $\overrightarrow{v(\gamma_f g)v}$ . Thus

$$\begin{split} \psi(\gamma_f g) &= a_{x_2} \,\lambda_{f_2} \dots a_{x_r} \,\lambda_{f_r} \,a_v R \\ &= \lambda_f \,\lambda_{\bar{f}} \,a_{x_2} \,\lambda_{f_2} \dots a_{x_r} \,\lambda_{f_r} \,a_v R \\ &= \psi(\gamma_f) \psi(g) \,. \end{split}$$

This completes the proof that  $\psi$  is a homomorphism.

Our construction of  $\psi$  ensures that if  $\psi(g) = R$  then g = 1. So  $\psi$  is injective. The cosets  $h_w R$  (w a vertex of T and h(w) = w) and  $\lambda_f R$  (f an edge of X/G) together generate  $[(*G_w)*F]/R$ . Now  $\psi(h) = h_x R$  where x is the nearest fixed point of h to v. But h fixes all of  $\overline{xw}$  so

$$\psi(h) = h_{\rm x}R = h_{\rm w}R \,.$$

Also

$$\psi(\gamma_f) = \lambda_f R \, .$$

Therefore the image of  $\psi$  is all of  $[(*G_w)*F]/R$  and we have shown that  $\psi$  is an isomorphism.

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