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is an isomorphism.

In particular if $\operatorname{Supp} \mathcal{M} \subset \Lambda_1 \cup \Lambda_2$ and if $\dim (\Lambda_1 \cap \Lambda_2) \leqslant n-2$, then \mathcal{M} is a direct sum of two holonomic \mathscr{E}_X -modules supported on Λ_1 and Λ_2 , respectively.

Here is another type of theorem.

Theorem 10.4.3 ([SKKO]). Let $\mathcal{M}=\mathcal{E}u=\mathcal{E}/\mathcal{J}$ be a holonomic \mathcal{E} -module defined on a neighborhood of $p\in T^*X$. Assume Supp $\mathcal{M}=\Lambda_1\cup\Lambda_2$ and

- (i) Λ_1 , Λ_2 and $\Lambda_1 \cap \Lambda_2$ are non-singular and $\dim \Lambda_1 = \dim \Lambda_2 = n$, $\dim (\Lambda_1 \cap \Lambda_2) = n 1$.
- (ii) $T_{p'}\Lambda_1\cap T_{p'}\Lambda_2=T_{p'}(\Lambda_1\cap\Lambda_2)$ for any p' in a neighborhood of p in $\Lambda_1\cap\Lambda_2$.
- (iii) The symbol ideal of $\mathscr I$ coincides with the ideal of functions vanishing on $\Lambda_1 \cup \Lambda_2$.

Setting $k = \operatorname{ord}_{\Lambda_1} u - \operatorname{ord}_{\Lambda_2} u - 1/2$, we have

- (a) \mathcal{M} has a non-zero quotient supported on $\Lambda_1 \Leftrightarrow \mathcal{M}$ has a non-zero submodule supported on $\Lambda_2 \Leftrightarrow k \in \mathbf{Z}$.
- (b) \mathcal{M}_p is a simple \mathcal{E}_p -module $\Leftrightarrow k \notin \mathbb{Z}$.

Sketch of the proof. By a quantized contact transformation, we can transform p, Λ_1, Λ_2 and $\mathcal J$ as follows:

$$\begin{split} p &= (0, dx_1) \\ \Lambda_1 &= \{ (x, \xi); x_1 = \xi_2 = \dots = \xi_n = 0 \} \\ \Lambda_2 &= \{ (x, \xi); x_1 = x_2 = \xi_3 = \dots = \xi_n = 0 \} \\ \mathscr{I} &= \mathscr{E}(x_1 \partial_1 - \lambda) + \mathscr{E}(x_2 \partial_2 - \mu) + \sum_{j \geq 2} \mathscr{E} \partial_j \end{split}$$

In this case, we can easily check the theorem.

§ 11. Application to the b-function (see [SKKO])

11.1. As one of the most successful application of microlocal analysis, we shall sketch here how to calculate the b-function of a function under certain conditions.

- 11.2. Let f be a holomorphic function on a complex manifold X. Then, it is proved ([Bj], [Be] [K1]) that there exist (locally) a non zero polynomial b(s) and $P(s) \in \mathcal{D}[s] = \mathcal{D} \otimes \mathbf{C}[s]$ such that $P(s)f(x)^{s+1} = b(s)f(x)^s$ for any $s \in \mathbb{N}$. Such a polynomial b(s) of smallest degree is called the b-function of f(x) and is denoted by $b_f(s)$. For the relations between the b-function and the local monodromy see [M1], [K3].
- 11.3. Set $\mathscr{J} = \{P(s) \in \mathscr{D}[s]; P(s)f^s = 0 \text{ for } s \in \mathbb{N}\}$ and $\mathscr{N} = \mathscr{D}[s]/\mathscr{J}$. We shall denote the canonical generator of \mathscr{N} by f^s . Then $t: \mathscr{N} \ni P(s)f^s \to P(s+1)f \cdot f^s \in \mathscr{N}$ gives a \mathscr{D} -endomorphism of \mathscr{N} and $t\mathscr{N} = \mathscr{D}[s]f^{s+1}$. Here $f^{s+1} = f \cdot f^s \in \mathscr{N}$. In this terminology $b_f(s)$ is the minimal polynomial of $s \in \mathscr{E}nd_{\mathscr{D}}(\mathscr{N}/t\mathscr{N})$.

For $\lambda \in \mathbb{C}$, we set $\mathcal{M}_{\lambda} = \mathcal{D}[s]/(\mathcal{J} + \mathcal{D}[s](s-\lambda))$ and denote by f^{λ} the canonical generator of \mathcal{M}_{λ} . Then $f^{\lambda+1} \mapsto f f^{\lambda}$ defines a \mathcal{D} -linear homomorphism $\mathcal{M}_{\lambda+1} \to \mathcal{M}_{\lambda}$.

11.4. Let W be the closure of

$$\{(s, x, \xi) \in \mathbb{C} \times T^*X; \xi = sd \log f(x), f(x) \neq 0\}$$

in $\mathbb{C} \times T^*X$. Set $W_0 = W \cap \{s=0\} \subset T^*X$. Then we can prove

PROPOSITION 11.4.1 ([K1]).

- (i) N is a coherent \mathcal{D}_X -module and $\mathrm{Ch}\,(\mathcal{N})=p(W)$, where p is the projection from $\mathbb{C}\times T^*X$ to T^*X .
- (ii) For any $\lambda \in \mathbb{C}$, \mathcal{M}_{λ} is a regular holonomic \mathcal{D}_{X} -module and $\mathrm{Ch}\,(\mathcal{M}_{\lambda}) = W_{0}$.
- (iii) $\mathcal{N}/t\mathcal{N}$ is a regular holonomic \mathcal{D}_X -module and $\operatorname{Ch}(\mathcal{N}/t\mathcal{N})$ = $W_0 \cap (\pi \circ f)^{-1}(0)$.
- 11.5. In the sequel, for the sake of simplicity, we assume that there exists a vector field v such that v(f) = f. Therefore we have $v^k(f^s) = s^k f^s$. Hence \mathcal{N} is a \mathcal{D} -module generated by f^s . If we set $\widetilde{\mathcal{J}} = \mathcal{D} \cap \mathcal{J}$, then $\mathcal{N} \cong \mathcal{D}/\mathcal{J}$ and $\mathcal{J} = \mathcal{D}[s](s-v) + \mathcal{D}[s]\widetilde{\mathcal{J}}$.
- 11.6. The following lemma is almost obvious but affords a fundamental tool to calculate the b-function.

LEMMA 11.6.1. Let \mathcal{L} be an \mathcal{E}_X -module and w a non-zero section of \mathcal{L} . For $\lambda \in \mathbb{C}$, we assume

- (i) $v(w) = \lambda w$
- (ii) $\mathcal{J}w = 0$
- (iii) fw = 0.

Then we have $b_f(\lambda) = 0$.

Proof. There is a $P \in \mathcal{D}$ such that $b_f(s)f^s = Pf^{s+1}$. Hence $(b_f(v) - Pf)f^s = 0$, which implies $b_f(v) - Pf \in \mathcal{J}$. Since $b_f(v)w = b_f(\lambda)w$ we have

$$0 = (b_f(v) - Pf)w = b_f(\lambda)w.$$

This implies $b_f(\lambda) = 0$.

11.7. Let $\bar{\mathcal{J}}$ be the symbol ideal of $\tilde{\mathcal{J}}$. Then the zero set of $\bar{\mathcal{J}}$ is W, and the zero of $\bar{\mathcal{J}} + \mathcal{O}\sigma(v)$ is W_0 . Let Λ be an irreducible component of W_0 . If $\bar{\mathcal{J}} + \mathcal{O}_{T*X}\sigma(v)$ is a reduced ideal at a generic point p of Λ then we call Λ a good Lagrangean.

If Λ is a good Lagrangean, then W is non-singular on a neighborhood of a generic point p of Λ and $\sigma = \sigma(s)|_W$ has non zero-differential. Let $p: W \to X$ denote the projection. We define $m(\Lambda) \in \mathbb{N}$ as the degree of zero of $f \circ p$ along Λ , and set $f_{\Lambda} = (f \circ p/\sigma^{m(\Lambda)})|_{\Lambda}$. Let ω be the non-vanishing n-form on X. Then $(p*\omega) \wedge d\sigma$ is an (n+1)-form on W. Let $\mu(\Lambda)$ be the degree of zeros of $(p*\omega) \wedge d\sigma$ along Λ , and let η be the n-form on Λ given by

$$\left. \frac{p^* \omega \wedge d\sigma}{\sigma^{\mu(\Lambda)}} \right|_{\Lambda} = \eta \wedge d\sigma.$$

If we set $\kappa_{\Lambda} = \eta \otimes \omega^{\otimes -1} \in \omega_{\Lambda} \otimes \omega_{X}^{\otimes -1}$, then this is independent of the choice of ω . We have

PROPOSITION 11.7.1 ([SKKO]). If Λ is a good Lagrangean, then for any $\lambda \in \mathbb{C}$, \mathcal{M}_{λ} is a simple holonomic system on a neighborhood of a generic point p of Λ and we have

(i)
$$\sigma(f^{\lambda}) = f^{\lambda}_{\Lambda} \sqrt{\kappa_{\Lambda}}$$
.

In particular

ord
$$f^{\lambda} = -m(\Lambda)\lambda - \mu(\Lambda)/2$$
.

(ii) There exists a monic polynomial $b_\Lambda(s)$ of degree $m(\Lambda)$ and an invertible micro-differential operator P_Λ of order $m(\Lambda)$ such that

$$b_{\Lambda}(s)f^s = P_{\Lambda}f \cdot f^s$$
 in $\mathscr{E} \underset{\mathscr{D}}{\otimes} \mathscr{N}$
$$\sigma(P_{\Lambda})|_{\Lambda} = f_{\Lambda}^{-1}.$$

and

Remark that f_{Λ} and ω_{Λ} are homogeneous of degree $-m(\Lambda)$ and $-\mu(\Lambda)$, respectively.

Remark also that the minimal polynomial of $s \in \mathscr{E}nd_{\mathscr{E}}(\mathscr{E} \otimes \mathscr{N}/t\mathscr{N})|_{\Lambda}$ is $b_{\Lambda}(s)$. In fact, if $Pf^{s+1} = b(s)f^{s}$ in $\mathscr{E} \otimes \mathscr{N}$, then $(P \cdot P_{\Lambda}^{-1}b_{\Lambda}(s) - b(s))f^{s} = 0$. This implies that $P \cdot P_{\Lambda}^{-1}b_{\Lambda}(v) - b(v) \in \mathscr{E}\mathscr{J}$. Hence

$$\sigma(P \cdot P_{\Lambda}^{-1}b_{\Lambda}(v) - b(v))|_{W} = 0.$$

If ord $P \cdot P_{\Lambda}^{-1}b_{\Lambda}(v) = \text{ord } P > \text{deg } b$, then $\sigma(P)|_{W} = 0$. Therefore P = P' + P'' with $P'' \in \mathscr{E}\mathscr{J}$ and $\sigma(P') < \sigma(P)$. Hence $P'f^{s+1} = b(s)f^{s}$. Thus, we may assume ord $P \leq \text{deg } b$. Then

$$0 = \sigma(b(v) - P \cdot P_{\Lambda}^{-1} b_{\Lambda}(v))|_{W} = b(\sigma) - (\sigma(P)|_{W} f_{\Lambda} b_{\Lambda}(\sigma)).$$

This shows that b(s) is a multiple of $b_{\Lambda}(s)$.

COROLLARY 11.7.2. If every irreducible component of W_0 is good Lagrangean, then $b_f(s)$ is the least common multiple of the $b_{\Lambda}(s)$.

- 11.8. Let Λ_1 and Λ_2 be two good Lagrangeans. We assume the following conditions for a point $p \in \Lambda_1 \cap \Lambda_2$:
- (11.8.1) $\dim_p \Lambda_1 \cap \Lambda_2 = n-1$ and Λ_1 , Λ_2 and $\Lambda_1 \cap \Lambda_2$ are non singular on a neighborhood of p.
- (11.8.2) For any point p' on a neighborhood of p in $\Lambda_1 \cap \Lambda_2$, we have $T_{p'}\Lambda_1 \cap T_{p'}\Lambda_2 = T_{p'}(\Lambda_1 \cap \Lambda_2)$.
- (11.8.3) $\bar{\mathscr{J}} + \mathscr{O}\sigma(v)$ coincides with the defining ideal of $\Lambda_1 \cup \Lambda_2$ with the reduced structure.

In this case we say that Λ_1 and Λ_2 have a good intersection.

We have the following theorem.

Theorem 11.7.3. Let Λ_1 and Λ_2 be good Lagrangeans with a good intersection. If $m(\Lambda_1) \geqslant m(\Lambda_2)$, then

$$\prod_{k=0}^{m(\Lambda_1)-m(\Lambda_2)-1} \left(\operatorname{ord}_{\Lambda_2} f^s - \operatorname{ord}_{\Lambda_1} f^s + \frac{1}{2} + k \right) \left| b_f(s) \right|.$$

In order to prove this let us take $\lambda \in \mathbb{C}$ such that

$$k = \operatorname{ord}_{\Lambda_1} f^{\lambda} - \operatorname{ord}_{\Lambda_2} f^{\lambda} - 1/2 \in \mathbf{N} \quad \text{and}$$

$$(11.8.4)$$

$$k' = \operatorname{ord}_{\Lambda_1} f^{\lambda+1} - \operatorname{ord}_{\Lambda_2} f^{\lambda+1} - 1/2 \in \mathbf{N}.$$

Recall that

$$k = (m(\Lambda_2) - m(\Lambda_1))\lambda - \frac{1}{2}(\mu(\Lambda_2) - \mu(\Lambda_1) - 1/2)$$

and $k' = k + (m(\Lambda_2) - m(\Lambda_1))$. Then by Theorem 10.4.3, \mathcal{M}_{λ} has a non-zero quotient \mathcal{L} whose support is Λ_1 . Let $w \in \mathcal{L}$ be the image of $f^{\lambda} \in \mathcal{M}_{\lambda}$.

Let $\alpha \colon \mathscr{M}_{\lambda} \to \mathscr{L}$ be the canonical homomorphism and $\beta \colon \mathscr{M}_{\lambda+1} \to \mathscr{M}_{\lambda}$ be the homomorphism given by $f^{\lambda+1} \mapsto f \cdot f^{\lambda}$. Then, since $k' \notin \mathbb{N}$, $\mathscr{M}_{\lambda+1}$ has no non-zero quotient supported in Λ_1 . Hence $\alpha \circ \beta = 0$. Therefore $fw = \alpha \beta(f^{\lambda+1}) = 0$. Thus we can apply Lemma 11.6.1 to conclude that $b_f(\lambda) = 0$. If $k \in \mathbb{Z}$ with $0 \leqslant k < m(\Lambda_1) - m(\Lambda_2)$ then

$$\lambda = \frac{1}{m(\Lambda_1) - m(\Lambda_2)} \left(k + \frac{1}{2} \left(\mu(\Lambda_1) - \mu(\Lambda_2) - 1 \right) \right)$$

satisfies (11.8.4). This shows that $b_f(s)$ is a multiple of

$$\prod_{k=0}^{m(\Lambda_1)-m(\Lambda_2)-1} \left(\left(m(\Lambda_1) - m(\Lambda_2) \right) s - \frac{1}{2} \left(\mu(\Lambda_1) - \mu(\Lambda_2) - 1 \right) + k \right)$$

$$= \text{const.} \quad \prod_{k=0}^{m(\Lambda_1)-m(\Lambda_2)-1} \left(\text{ord}_{\Lambda_2} f^s - \text{ord}_{\Lambda_1} f^s + \frac{1}{2} + k \right).$$

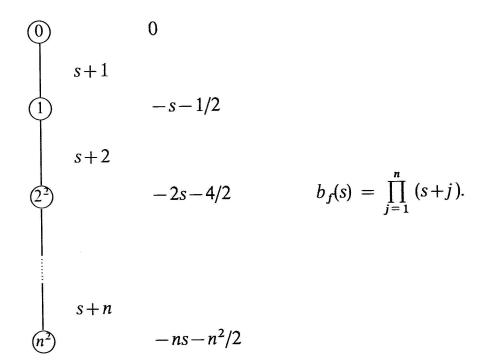
If we refine this argument, we can prove

Theorem 11.8.2 ([SKKO]). If Λ_1 and Λ_2 are good Lagrangeans with a good intersection and if $m(\Lambda_1) \geqslant m(\Lambda_2)$ then

$$\frac{b_{\Lambda_1}(s)}{b_{\Lambda_2}(s)} = \text{const.} \prod_{k=0}^{m(\Lambda_1)-m(\Lambda_2)-1} \left(\text{ord}_{\Lambda_2} f^s - \text{ord}_{\Lambda_1} f^s + \frac{1}{2} + k \right).$$

Example 11.8.3.

(i)
$$X = M_n(\mathbb{C}) = \mathbb{C}^{n^2}$$
 and $f(x) = \det x$.



Here (a) means a good Lagrangean which is the conormal bundle to an a-codimensional submanifold. — means that the two corresponding good Lagrangeans have a good intersection.

The polynomial attached to the intersection is the ratio of the corresponding b_{Λ} -functions, calculated by Theorem 11.8.2. The polynomial attached to the circle is the order of f^{λ} .

(ii)
$$X = \mathbb{C}^n$$
, $f(x) = x_1^2 + ... + x_n^2$

$$0$$

$$s+1$$

$$1$$

$$s+n/2$$

$$n$$

$$-2s-n/2$$

$$X = \mathbb{C}^3 \quad f = x^2y + z^2$$

$$0$$

$$b(s) = (s+1)(s+n/2)$$

$$-2s-n/2$$

(iii)
$$X = \mathbb{C}^3$$
 $f = x^2y + z^2$

