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is an isomorphism.

In particular if $\text{Supp } \mathcal{M} \subset \Lambda_1 \cup \Lambda_2$ and if $\dim(\Lambda_1 \cap \Lambda_2) \leq n-2$, then \mathcal{M} is a direct sum of two holonomic \mathcal{E}_X -modules supported on Λ_1 and Λ_2 , respectively.

Here is another type of theorem.

THEOREM 10.4.3 ([SKKO]). Let $\mathcal{M} = \mathcal{E}u = \mathcal{E}/\mathcal{J}$ be a holonomic \mathcal{E} -module defined on a neighborhood of $p \in T^*X$. Assume $\text{Supp } \mathcal{M} = \Lambda_1 \cup \Lambda_2$ and

- (i) Λ_1, Λ_2 and $\Lambda_1 \cap \Lambda_2$ are non-singular and $\dim \Lambda_1 = \dim \Lambda_2 = n, \dim(\Lambda_1 \cap \Lambda_2) = n-1$.
- (ii) $T_{p'} \Lambda_1 \cap T_{p'} \Lambda_2 = T_{p'}(\Lambda_1 \cap \Lambda_2)$ for any p' in a neighborhood of p in $\Lambda_1 \cap \Lambda_2$.
- (iii) The symbol ideal of \mathcal{J} coincides with the ideal of functions vanishing on $\Lambda_1 \cup \Lambda_2$.

Setting $k = \text{ord}_{\Lambda_1} u - \text{ord}_{\Lambda_2} u - 1/2$, we have

- (a) \mathcal{M} has a non-zero quotient supported on $\Lambda_1 \Leftrightarrow \mathcal{M}$ has a non-zero submodule supported on $\Lambda_2 \Leftrightarrow k \in \mathbb{Z}$.
- (b) \mathcal{M}_p is a simple \mathcal{E}_p -module $\Leftrightarrow k \notin \mathbb{Z}$.

Sketch of the proof. By a quantized contact transformation, we can transform p, Λ_1, Λ_2 and \mathcal{J} as follows:

$$p = (0, dx_1)$$

$$\Lambda_1 = \{(x, \xi); x_1 = \xi_2 = \dots = \xi_n = 0\}$$

$$\Lambda_2 = \{(x, \xi); x_1 = x_2 = \xi_3 = \dots = \xi_n = 0\}$$

$$\mathcal{J} = \mathcal{E}(x_1 \partial_1 - \lambda) + \mathcal{E}(x_2 \partial_2 - \mu) + \sum_{j \geq 2} \mathcal{E} \partial_j$$

In this case, we can easily check the theorem.

§ 11. APPLICATION TO THE b -FUNCTION (see [SKKO])

11.1. As one of the most successful application of microlocal analysis, we shall sketch here how to calculate the b -function of a function under certain conditions.

11.2. Let f be a holomorphic function on a complex manifold X . Then, it is proved ([Bj], [Be] [K1]) that there exist (locally) a non zero polynomial $b(s)$ and $P(s) \in \mathcal{D}[s] = \mathcal{D} \underset{\text{def}}{\otimes} \underset{\text{C}}{\mathbb{C}}[s]$ such that $P(s)f(x)^{s+1} = b(s)f(x)^s$ for any $s \in \mathbb{N}$. Such a polynomial $b(s)$ of smallest degree is called the b -function of $f(x)$ and is denoted by $b_f(s)$. For the relations between the b -function and the local monodromy see [M1], [K3].

11.3. Set $\mathcal{J} = \{P(s) \in \mathcal{D}[s]; P(s)f^s = 0 \text{ for } s \in \mathbb{N}\}$ and $\mathcal{N} = \mathcal{D}[s]/\mathcal{J}$. We shall denote the canonical generator of \mathcal{N} by f^s . Then $t: \mathcal{N} \ni P(s)f^s \rightarrow P(s+1)f \cdot f^s \in \mathcal{N}$ gives a \mathcal{D} -endomorphism of \mathcal{N} and $t\mathcal{N} = \mathcal{D}[s]f^{s+1}$. Here $f^{s+1} = f \cdot f^s \in \mathcal{N}$. In this terminology $b_f(s)$ is the minimal polynomial of $s \in \mathcal{E}nd_{\mathcal{D}}(\mathcal{N}/t\mathcal{N})$.

For $\lambda \in \mathbb{C}$, we set $\mathcal{M}_\lambda = \mathcal{D}[s]/(\mathcal{J} + \mathcal{D}[s](s-\lambda))$ and denote by f^λ the canonical generator of \mathcal{M}_λ . Then $f^{\lambda+1} \mapsto f f^\lambda$ defines a \mathcal{D} -linear homomorphism $\mathcal{M}_{\lambda+1} \rightarrow \mathcal{M}_\lambda$.

11.4. Let W be the closure of

$$\{(s, x, \xi) \in \mathbb{C} \times T^*X; \xi = sd \log f(x), f(x) \neq 0\}$$

in $\mathbb{C} \times T^*X$. Set $W_0 = W \cap \{s=0\} \subset T^*X$. Then we can prove

PROPOSITION 11.4.1 ([K1]).

- (i) N is a coherent \mathcal{D}_X -module and $\text{Ch}(\mathcal{N}) = p(W)$, where p is the projection from $\mathbb{C} \times T^*X$ to T^*X .
- (ii) For any $\lambda \in \mathbb{C}$, \mathcal{M}_λ is a regular holonomic \mathcal{D}_X -module and $\text{Ch}(\mathcal{M}_\lambda) = W_0$.
- (iii) $\mathcal{N}/t\mathcal{N}$ is a regular holonomic \mathcal{D}_X -module and $\text{Ch}(\mathcal{N}/t\mathcal{N}) = W_0 \cap (\pi_0 f)^{-1}(0)$.

11.5. In the sequel, for the sake of simplicity, we assume that there exists a vector field v such that $v(f) = f$. Therefore we have $v^k(f^s) = s^k f^s$. Hence \mathcal{N} is a \mathcal{D} -module generated by f^s . If we set $\tilde{\mathcal{J}} = \mathcal{D} \cap \mathcal{J}$, then $\mathcal{N} \cong \mathcal{D}/\tilde{\mathcal{J}}$ and $\mathcal{J} = \mathcal{D}[s](s-v) + \mathcal{D}[s]\tilde{\mathcal{J}}$.

11.6. The following lemma is almost obvious but affords a fundamental tool to calculate the b -function.

LEMMA 11.6.1. Let \mathcal{L} be an \mathcal{E}_X -module and w a non-zero section of \mathcal{L} . For $\lambda \in \mathbb{C}$, we assume

- (i) $v(w) = \lambda w$
- (ii) $\tilde{\mathcal{J}}w = 0$
- (iii) $fw = 0$.

Then we have $b_f(\lambda) = 0$.

Proof. There is a $P \in \mathcal{D}$ such that $b_f(s)f^s = Pf^{s+1}$. Hence $(b_f(v) - Pf)f^s = 0$, which implies $b_f(v) - Pf \in \tilde{\mathcal{J}}$. Since $b_f(v)w = b_f(\lambda)w$ we have

$$0 = (b_f(v) - Pf)w = b_f(\lambda)w.$$

This implies $b_f(\lambda) = 0$.

11.7. Let $\tilde{\mathcal{J}}$ be the symbol ideal of $\tilde{\mathcal{J}}$. Then the zero set of $\tilde{\mathcal{J}}$ is W , and the zero of $\tilde{\mathcal{J}} + \mathcal{O}\sigma(v)$ is W_0 . Let Λ be an irreducible component of W_0 . If $\tilde{\mathcal{J}} + \mathcal{O}_{T^*X}\sigma(v)$ is a reduced ideal at a generic point p of Λ then we call Λ a *good Lagrangean*.

If Λ is a good Lagrangean, then W is non-singular on a neighborhood of a generic point p of Λ and $\sigma = \sigma(s)|_W$ has non zero-differential. Let $p: W \rightarrow X$ denote the projection. We define $m(\Lambda) \in \mathbb{N}$ as the degree of zero of $f \circ p$ along Λ , and set $f_\Lambda = (f \circ p / \sigma^{m(\Lambda)})|_\Lambda$. Let ω be the non-vanishing n -form on X . Then $(p^*\omega) \wedge d\sigma$ is an $(n+1)$ -form on W . Let $\mu(\Lambda)$ be the degree of zeros of $(p^*\omega) \wedge d\sigma$ along Λ , and let η be the n -form on Λ given by

$$\left. \frac{p^*\omega \wedge d\sigma}{\sigma^{m(\Lambda)}} \right|_\Lambda = \eta \wedge d\sigma.$$

If we set $\kappa_\Lambda = \eta \otimes \omega^{\otimes -1} \in \omega_\Lambda \otimes \omega_X^{\otimes -1}$, then this is independent of the choice of ω . We have

PROPOSITION 11.7.1 ([SKKO]). If Λ is a good Lagrangean, then for any $\lambda \in \mathbb{C}$, \mathcal{M}_λ is a simple holonomic system on a neighborhood of a generic point p of Λ and we have

- (i) $\sigma(f^\lambda) = f_\Lambda^\lambda \sqrt{\kappa_\Lambda}$.

In particular

$$\text{ord } f^\lambda = -m(\Lambda)\lambda - \mu(\Lambda)/2.$$

- (ii) There exists a monic polynomial $b_\Lambda(s)$ of degree $m(\Lambda)$ and an invertible micro-differential operator P_Λ of order $m(\Lambda)$ such that

$$b_\Lambda(s)f^s = P_\Lambda f \cdot f^s \quad \text{in} \quad \mathcal{E} \underset{\mathcal{D}}{\otimes} \mathcal{N}$$

and
$$\sigma(P_\Lambda)|_\Lambda = f_\Lambda^{-1}.$$

Remark that f_Λ and ω_Λ are homogeneous of degree $-m(\Lambda)$ and $-\mu(\Lambda)$, respectively.

Remark also that the minimal polynomial of $s \in \mathcal{E}nd_{\mathcal{E}}(\mathcal{E} \underset{\mathcal{D}}{\otimes} \mathcal{N}/t\mathcal{N})|_\Lambda$ is $b_\Lambda(s)$. In fact, if $Pf^{s+1} = b(s)f^s$ in $\mathcal{E} \otimes \mathcal{N}$, then $(P \cdot P_\Lambda^{-1}b_\Lambda(s) - b(s))f^s = 0$. This implies that $P \cdot P_\Lambda^{-1}b_\Lambda(v) - b(v) \in \mathcal{E}\tilde{\mathcal{J}}$. Hence

$$\sigma(P \cdot P_\Lambda^{-1}b_\Lambda(v) - b(v))|_W = 0.$$

If $\text{ord } P \cdot P_\Lambda^{-1}b_\Lambda(v) = \text{ord } P > \deg b$, then $\sigma(P)|_W = 0$. Therefore $P = P' + P''$ with $P'' \in \mathcal{E}\tilde{\mathcal{J}}$ and $\sigma(P') < \sigma(P)$. Hence $P'f^{s+1} = b(s)f^s$. Thus, we may assume $\text{ord } P \leq \deg b$. Then

$$0 = \sigma(b(v) - P \cdot P_\Lambda^{-1}b_\Lambda(v))|_W = b(\sigma) - (\sigma(P)|_W f_\Lambda b_\Lambda(\sigma)).$$

This shows that $b(s)$ is a multiple of $b_\Lambda(s)$.

COROLLARY 11.7.2. *If every irreducible component of W_0 is good Lagrangean, then $b_f(s)$ is the least common multiple of the $b_\Lambda(s)$.*

11.8. Let Λ_1 and Λ_2 be two good Lagrangeans. We assume the following conditions for a point $p \in \Lambda_1 \cap \Lambda_2$:

- (11.8.1) $\dim_p \Lambda_1 \cap \Lambda_2 = n-1$ and Λ_1 , Λ_2 and $\Lambda_1 \cap \Lambda_2$ are non singular on a neighborhood of p .
- (11.8.2) For any point p' on a neighborhood of p in $\Lambda_1 \cap \Lambda_2$, we have $T_{p'}\Lambda_1 \cap T_{p'}\Lambda_2 = T_{p'}(\Lambda_1 \cap \Lambda_2)$.
- (11.8.3) $\tilde{\mathcal{J}} + \mathcal{O}\sigma(v)$ coincides with the defining ideal of $\Lambda_1 \cup \Lambda_2$ with the reduced structure.

In this case we say that Λ_1 and Λ_2 have a *good intersection*.

We have the following theorem.

THEOREM 11.7.3. Let Λ_1 and Λ_2 be good Lagrangeans with a good intersection. If $m(\Lambda_1) \geq m(\Lambda_2)$, then

$$\prod_{k=0}^{m(\Lambda_1)-m(\Lambda_2)-1} \left(\text{ord}_{\Lambda_2} f^s - \text{ord}_{\Lambda_1} f^s + \frac{1}{2} + k \right) \mid b_f(s).$$

In order to prove this let us take $\lambda \in \mathbb{C}$ such that

$$(11.8.4) \quad \begin{aligned} k &= \text{ord}_{\Lambda_1} f^\lambda - \text{ord}_{\Lambda_2} f^\lambda - 1/2 \in \mathbb{N} \quad \text{and} \\ k' &= \text{ord}_{\Lambda_1} f^{\lambda+1} - \text{ord}_{\Lambda_2} f^{\lambda+1} - 1/2 \in \mathbb{N}. \end{aligned}$$

Recall that

$$k = (m(\Lambda_2) - m(\Lambda_1))\lambda - \frac{1}{2}(\mu(\Lambda_2) - \mu(\Lambda_1) - 1/2)$$

and $k' = k + (m(\Lambda_2) - m(\Lambda_1))$. Then by Theorem 10.4.3, \mathcal{M}_λ has a non-zero quotient \mathcal{L} whose support is Λ_1 . Let $w \in \mathcal{L}$ be the image of $f^\lambda \in \mathcal{M}_\lambda$.

Let $\alpha: \mathcal{M}_\lambda \rightarrow \mathcal{L}$ be the canonical homomorphism and $\beta: \mathcal{M}_{\lambda+1} \rightarrow \mathcal{M}_\lambda$ be the homomorphism given by $f^{\lambda+1} \mapsto f \cdot f^\lambda$. Then, since $k' \notin \mathbb{N}$, $\mathcal{M}_{\lambda+1}$ has no non-zero quotient supported in Λ_1 . Hence $\alpha \circ \beta = 0$. Therefore $f w = \alpha \beta(f^{\lambda+1}) = 0$. Thus we can apply Lemma 11.6.1 to conclude that $b_f(\lambda) = 0$. If $k \in \mathbb{Z}$ with $0 \leq k < m(\Lambda_1) - m(\Lambda_2)$ then

$$\lambda = \frac{1}{m(\Lambda_1) - m(\Lambda_2)} \left(k + \frac{1}{2}(\mu(\Lambda_1) - \mu(\Lambda_2) - 1) \right)$$

satisfies (11.8.4). This shows that $b_f(s)$ is a multiple of

$$\begin{aligned} & \prod_{k=0}^{m(\Lambda_1)-m(\Lambda_2)-1} \left((m(\Lambda_1) - m(\Lambda_2))s - \frac{1}{2}(\mu(\Lambda_1) - \mu(\Lambda_2) - 1) + k \right) \\ &= \text{const.} \prod_{k=0}^{m(\Lambda_1)-m(\Lambda_2)-1} \left(\text{ord}_{\Lambda_2} f^s - \text{ord}_{\Lambda_1} f^s + \frac{1}{2} + k \right). \end{aligned}$$

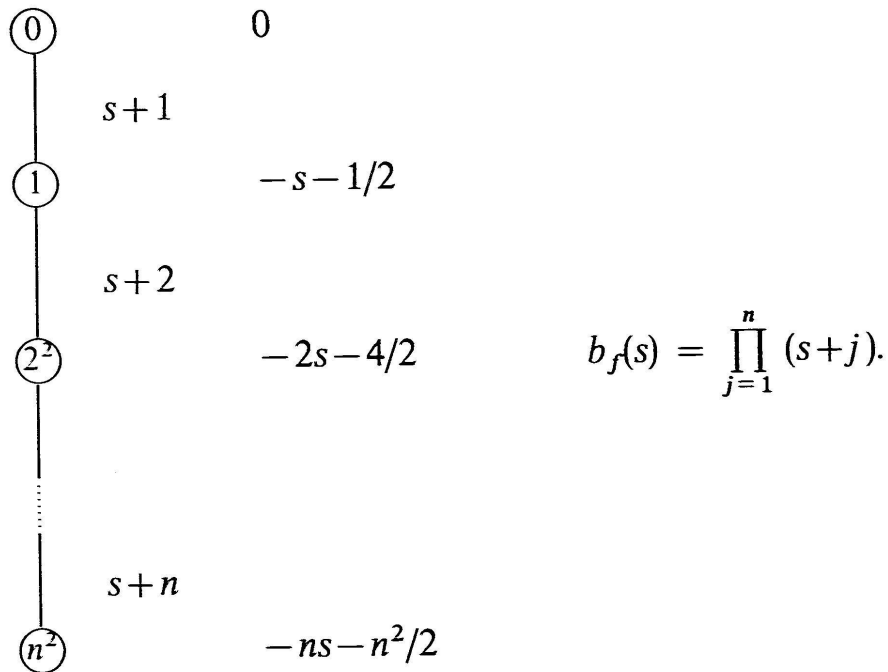
If we refine this argument, we can prove

THEOREM 11.8.2 ([SKKO]). If Λ_1 and Λ_2 are good Lagrangeans with a good intersection and if $m(\Lambda_1) \geq m(\Lambda_2)$ then

$$\frac{b_{\Lambda_1}(s)}{b_{\Lambda_2}(s)} = \text{const.} \prod_{k=0}^{m(\Lambda_1)-m(\Lambda_2)-1} \left(\text{ord}_{\Lambda_2} f^s - \text{ord}_{\Lambda_1} f^s + \frac{1}{2} + k \right).$$

Example 11.8.3.

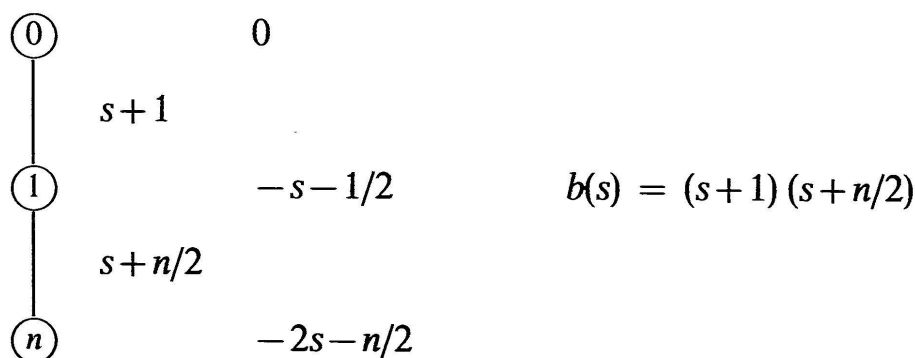
(i) $X = M_n(\mathbb{C}) = \mathbb{C}^{n^2}$ and $f(x) = \det x$.



Here \textcircled{a} means a good Lagrangean which is the conormal bundle to an a -codimensional submanifold. $\textcircled{a} - \textcircled{b}$ means that the two corresponding good Lagrangeans have a good intersection.

The polynomial attached to the intersection is the ratio of the corresponding b_Λ -functions, calculated by Theorem 11.8.2. The polynomial attached to the circle is the order of f^λ .

(ii) $X = \mathbb{C}^n, f(x) = x_1^2 + \dots + x_n^2$



(iii) $X = \mathbb{C}^3, f = x^2y + z^2$

