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9.4. Noting that any nowhere dense closed analytic subset of a Lagrangean variety is never involutive, Theorem 9.2.3 implies the following theorem.

**THEOREM 9.4.1.** *Let  $\mathcal{M}$  be a holonomic  $\mathcal{E}_X$ -module. Then the following conditions are equivalent.*

- (i) *There exists a Lagrangean subvariety  $\Lambda$  such that  $\mathcal{M}$  has regular singularities along  $\Lambda$ .*
- (ii) *For any involutive subvariety  $\Lambda$  which contains  $\text{Supp } \mathcal{M}$ ,  $\mathcal{M}$  has regular singularities along  $\Lambda$ .*
- (iii) *There exists an open dense subset  $\Omega$  of  $\text{Supp } \mathcal{M}$  such that  $\mathcal{M}$  has regular singularities along  $\text{Supp } \mathcal{M}$  on  $\Omega$ .*

If these equivalent conditions are satisfied, we say that  $\mathcal{M}$  is a *regular holonomic  $\mathcal{E}_X$ -module*.

The following properties are almost immediate.

**THEOREM 9.4.2.**

- (i) *Let  $0 \rightarrow \mathcal{M}' \rightarrow \mathcal{M} \rightarrow \mathcal{M}'' \rightarrow 0$  be an exact sequence of three coherent  $\mathcal{E}_X$ -modules. If two of them are regular holonomic then so is the third.*
- (ii) *If  $\mathcal{M}$  is regular holonomic, its dual  $\mathcal{M}^*$  is also regular holonomic.*

We just mention another analytic property of regular holonomic modules, which generalizes the fact that a formal solution of an ordinary differential equation with regular singularity converges.

**THEOREM 9.4.3** ([KK] Theorem 6.1.3). *If  $\mathcal{M}$  and  $\mathcal{N}$  are regular holonomic  $\mathcal{E}_X$ -modules, then  $\mathcal{E}xt_{\mathcal{E}_X}^j(\mathcal{M}, \mathcal{N}) \rightarrow \mathcal{E}xt_{\mathcal{E}_X}^j(\mathcal{M}, \widehat{\mathcal{E}}_X \otimes_{\mathcal{E}_X} \mathcal{N})$  and  $\mathcal{E}xt_{\mathcal{E}_X}^j(\mathcal{M}, \mathcal{N}) \rightarrow \mathcal{E}xt_{\mathcal{E}_X}^j(\mathcal{M}, \mathcal{E}_X^\infty \otimes_{\mathcal{E}_X} \mathcal{N})$  are isomorphisms.*

## § 10. STRUCTURE OF REGULAR HOLONOMIC $\mathcal{E}$ -MODULES

(See [SKK], [KK])

10.1. Let  $\Lambda$  be a Lagrangean submanifold of  $T^*X$ . We define  $\mathcal{I}_\Lambda$  and  $\mathcal{E}_\Lambda$  as in § 9.2.

Then  $\mathcal{E}_\Lambda(-1) = \mathcal{E}_\Lambda \cdot \mathcal{E}(-1)$  is a two-sided ideal of  $\mathcal{E}_\Lambda$  and  $\mathcal{E}_\Lambda/\mathcal{E}_\Lambda(-1)$  is a sheaf of rings which contains  $\mathcal{O}_\Lambda(0) = \mathcal{E}(0)/\mathcal{I}_\Lambda(-1)$ , the sheaf of homogeneous functions on  $\Lambda$ .

Let us take an invertible  $\mathcal{O}_\Lambda$ -module  $\mathcal{L}$  such that  $\mathcal{L}^{\otimes 2} \cong \omega_\Lambda \otimes_{\theta_X} \omega_X^{\otimes -1}$ . Such an  $\mathcal{L}$  exists at least locally. For  $P = P_1(x, \partial) + P_0(x, \partial) + \dots \in \mathcal{I}$  we define, for  $\varphi \in \mathcal{O}_\Lambda$  and an invertible section  $s$  of  $\mathcal{L}$ ,

$$L(P)(\varphi s) = \left\{ H_{P_1}(\varphi) + \frac{1}{2} \varphi \frac{L_{H_{P_1}}(s^{\otimes 2} \otimes dx)}{s^{\otimes 2} \otimes dx} + \left( P_0 - \frac{1}{2} \sum_{j=1}^n \frac{\partial^2 P_1}{\partial x_i \partial \xi_i} \right) \varphi \right\} s.$$

Here  $dx = dx_1 \wedge \dots \wedge dx_n \in \omega_X$  and  $s^{\otimes 2} \otimes dx$  is regarded as a section of  $\omega_\Lambda$ . The Lie derivative  $L_{H_{P_1}}$  of  $H_{P_1}$  operates on  $\omega_\Lambda$  as the first order differential operators so that  $L_{H_{P_1}}(s^{\otimes 2} \otimes dx)$  is a section of  $\omega_\Lambda$  and  $L_{H_{P_1}}(s^{\otimes 2} \otimes dx)/s^{\otimes 2} \otimes dx$  is a function on  $\Lambda$ .

We thus obtain  $L: \mathcal{I}_\Lambda \rightarrow \mathcal{E}nd_{\mathcal{C}}(\mathcal{L})$ . Then this does not depend on the choice of local coordinate system and moreover it extends to the ring homomorphism  $L: \mathcal{E}_\Lambda \rightarrow \mathcal{E}nd_{\mathcal{C}}(\mathcal{L})$ . Since the image is contained in the differential endomorphism of  $\mathcal{L}$ , we obtain the ring homomorphism  $L: \mathcal{E}_\Lambda \rightarrow \mathcal{L} \otimes_{\mathcal{O}_\Lambda} \mathcal{D}_\Lambda \otimes_{\mathcal{O}_\Lambda} \mathcal{L}^{\otimes -1}$ .

**PROPOSITION 10.1.1.** *By  $L, \mathcal{E}_\Lambda/\mathcal{E}_\Lambda(-1)$  coincides with the subsheaf of  $\mathcal{L} \otimes_{\mathcal{O}_\Lambda} \mathcal{D}_\Lambda \otimes_{\mathcal{O}_\Lambda} \mathcal{L}^{\otimes -1}$  consisting of differential endomorphisms of  $\mathcal{L}$  homogeneous of degree 0.*

If we take

$$\mathcal{I}_\Lambda \in \mathfrak{D} = \mathfrak{D}_1(x, \partial) + \mathfrak{D}_0(x, \partial) + \dots$$

such that  $d\mathfrak{D}_1 \equiv -\theta_X \text{ mod } I_\Lambda \Omega^1$  and

$$\frac{1}{2} \sum \frac{\partial^2 \mathfrak{D}_1}{\partial x_i \partial \xi_i} \equiv \mathfrak{D}_0(x, \xi) \text{ mod } \mathcal{I}_\Lambda$$

then  $L(\mathfrak{D})$  gives the Euler operator of  $\mathcal{L}$ . Such a  $\mathfrak{D}$  is unique modulo  $\mathcal{I}_\Lambda^2(-1) = \mathcal{E}_\Lambda(-1) \cap \mathcal{E}_X(1)$ .

10.2. Let  $\mathcal{M}$  be a regular holonomic  $\mathcal{E}_X$ -module whose support is  $\Lambda$ . Let  $\mathcal{M}_0$  be a coherent sub- $\mathcal{E}_\Lambda$ -module of  $\mathcal{M}$  which generates  $\mathcal{M}$ . Such an  $\mathcal{M}_0$  is called a *saturated lattice* of  $\mathcal{M}$ . Then  $\bar{\mathcal{M}} = \mathcal{M}_0/\mathcal{E}(-1)\mathcal{M}_0$  is an  $\mathcal{E}_\Lambda/\mathcal{E}_\Lambda(-1)$ -module, which is coherent over  $\mathcal{O}_\Lambda(0)$ .

Since a coherent sheaf with integrable connection is locally free, we have

**LEMMA 10.2.1.**  *$\bar{\mathcal{M}}$  is a locally free  $\mathcal{O}_\Lambda(0)$ -module of finite rank.*

Since  $\mathfrak{g}$  belongs to the center of  $\mathcal{E}_\Lambda/\mathcal{E}_\Lambda(-1)$ ,  $\mathfrak{g}$  can be considered as an endomorphism of  $\mathcal{H}om_{\mathcal{E}_\Lambda/\mathcal{E}_\Lambda(-1)}(\bar{\mathcal{M}}, \mathcal{L})$ , which is a locally constant sheaf on  $\Lambda$ . Its eigenvalues are called the *order* of  $\mathcal{M}$  with respect to  $\mathcal{M}_0$ .

10.3. Let us take a section  $G \subset \mathbf{C}$  of  $\mathbf{C} \rightarrow \mathbf{C}/\mathbf{Z}$ . Then there exists a unique saturated lattice  $\mathcal{M}_0$  such that the orders of  $\mathcal{M}$  with respect to  $\mathcal{M}_0$  are contained in  $G$  (See [K4]). Then

$$\mathcal{F} = \mathcal{H}om_{\mathcal{E}_\Lambda/\mathcal{E}_\Lambda(-1)}(\bar{\mathcal{M}}, \mathcal{L})$$

and

$$M = \exp 2\pi i \mathfrak{g} \in \mathcal{A}ut(\mathcal{F})$$

does not depend on the choice of  $G$ .

**THEOREM 10.3.1** ([KK] Chapter I, § 3). *Assume that there exists an invertible  $\mathcal{O}_\Lambda$ -module  $\mathcal{L}$  such that  $\mathcal{L}^{\otimes 2} = \omega_\Lambda \otimes \omega_X^{\otimes -1}$ . Then the category of regular holonomic  $\mathcal{E}_X$ -modules with support in  $\Lambda$  is equivalent to the category of  $(\mathcal{F}, M)$ 's where  $\mathcal{F}$  is a locally constant  $\mathbf{C}_\Lambda$ -module and  $M \in \mathcal{A}ut_{\mathbf{C}}(\mathcal{F})$ .*

10.4. If  $u \in \mathcal{M}$ , then the solution to  $L(P)\varphi = 0$  for  $P \in \mathcal{E}_\Lambda$  with  $Pu = 0$  is called a principal symbol of  $u$  and denoted by  $\sigma(u)$ . The homogeneous degree of  $\sigma(u)$  is called the order of  $u$ . In the terminology of § 10.2, the principal symbol is a section of  $\mathcal{H}om_{\mathcal{E}_\Lambda/\mathcal{E}_\Lambda(-1)}(\mathcal{E}_\Lambda u/\mathcal{E}_\Lambda(-1)u, \mathcal{L})$  and the order is the eigenvalue of  $\mathfrak{g}$  in  $\mathcal{H}om_{\mathcal{E}_\Lambda/\mathcal{E}_\Lambda(-1)}(\mathcal{E}_\Lambda u/\mathcal{E}_\Lambda(-1)u, \mathcal{L})$ .

10.4. When the characteristic variety is not smooth, we don't know much about the structure of holonomic systems. In this direction, we have

**THEOREM 10.4.1** ([K-K] Theorem 1.2.2). *Let  $Z$  be a closed analytic subset of an open subset  $\Omega$  of  $T^*X$ ,  $n = \dim X$ , and let  $\mathcal{M}$  and  $\mathcal{N}$  be holonomic  $\mathcal{E}_X|_\Omega$ -modules.*

(i) *If  $\dim Z \leq n-1$ , then*

$$\Gamma(\Omega; \mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{N})) \rightarrow \Gamma(\Omega \setminus Z; \mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{N}))$$

*is injective.*

(ii) *If  $\dim Z \leq n-2$ , then*

$$\Gamma(\Omega; \mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{N})) \rightarrow \Gamma(\Omega \setminus Z; \mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{N}))$$

is an isomorphism.

In particular if  $\text{Supp } \mathcal{M} \subset \Lambda_1 \cup \Lambda_2$  and if  $\dim(\Lambda_1 \cap \Lambda_2) \leq n-2$ , then  $\mathcal{M}$  is a direct sum of two holonomic  $\mathcal{E}_X$ -modules supported on  $\Lambda_1$  and  $\Lambda_2$ , respectively.

Here is another type of theorem.

**THEOREM 10.4.3 ([SKKO]).** Let  $\mathcal{M} = \mathcal{E}u = \mathcal{E}/\mathcal{I}$  be a holonomic  $\mathcal{E}$ -module defined on a neighborhood of  $p \in T^*X$ . Assume  $\text{Supp } \mathcal{M} = \Lambda_1 \cup \Lambda_2$  and

- (i)  $\Lambda_1, \Lambda_2$  and  $\Lambda_1 \cap \Lambda_2$  are non-singular and  $\dim \Lambda_1 = \dim \Lambda_2 = n, \dim(\Lambda_1 \cap \Lambda_2) = n-1$ .
- (ii)  $T_{p'} \Lambda_1 \cap T_{p'} \Lambda_2 = T_{p'}(\Lambda_1 \cap \Lambda_2)$  for any  $p'$  in a neighborhood of  $p$  in  $\Lambda_1 \cap \Lambda_2$ .
- (iii) The symbol ideal of  $\mathcal{I}$  coincides with the ideal of functions vanishing on  $\Lambda_1 \cup \Lambda_2$ .

Setting  $k = \text{ord}_{\Lambda_1} u - \text{ord}_{\Lambda_2} u - 1/2$ , we have

- (a)  $\mathcal{M}$  has a non-zero quotient supported on  $\Lambda_1 \Leftrightarrow \mathcal{M}$  has a non-zero submodule supported on  $\Lambda_2 \Leftrightarrow k \in \mathbf{Z}$ .
- (b)  $\mathcal{M}_p$  is a simple  $\mathcal{E}_p$ -module  $\Leftrightarrow k \notin \mathbf{Z}$ .

*Sketch of the proof.* By a quantized contact transformation, we can transform  $p, \Lambda_1, \Lambda_2$  and  $\mathcal{I}$  as follows:

$$p = (0, dx_1)$$

$$\Lambda_1 = \{(x, \xi); x_1 = \xi_2 = \dots = \xi_n = 0\}$$

$$\Lambda_2 = \{(x, \xi); x_1 = x_2 = \xi_3 = \dots = \xi_n = 0\}$$

$$\mathcal{I} = \mathcal{E}(x_1 \partial_1 - \lambda) + \mathcal{E}(x_2 \partial_2 - \mu) + \sum_{j>2} \mathcal{E} \partial_j$$

In this case, we can easily check the theorem.

## § 11. APPLICATION TO THE $b$ -FUNCTION (see [SKKO])

11.1. As one of the most successful application of microlocal analysis, we shall sketch here how to calculate the  $b$ -function of a function under certain conditions.