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THEOREM 8.10.1. Let U_X and U_Y be open subsets of T^*X and T^*Y , respectively. Let \mathcal{N} be a coherent $(\mathcal{O}_Y|_{U_Y})$ -module. Assume

(i) $\rho_g: \tilde{\omega}_g^{-1}(\text{Supp } \mathcal{N}) \cap \rho_g^{-1}(U_X) \rightarrow U_X$ is a finite morphism.

Then we have

(a) $\mathcal{T}or_j^{\tilde{\omega}_g^{-1}\mathcal{O}_Y}(\mathcal{O}_{X \leftarrow Y}, \tilde{\omega}_g^{-1}\mathcal{N}) = 0$ for $j \neq 0$.

(b) $\mathcal{M} = \rho_{g*}(\mathcal{O}_{X \leftarrow Y} \otimes_{\tilde{\omega}_g^{-1}\mathcal{O}_Y} \tilde{\omega}_g^{-1}\mathcal{N})|_{U_X}$ is a coherent $\mathcal{O}_X|_{U_X}$ -module.

(c) $\text{Supp } \mathcal{M} = \rho_g(\tilde{\omega}_g^{-1} \text{Supp } \mathcal{N} \cap U_X)$.

§ 9. REGULARITY CONDITIONS (See [KK], [K-O])

9.1. Let us recall the notion of regular singularity of ordinary differential equations. Let $P(x, \partial) = \sum_{j \leq m} a_j(x)\partial^j$ be a linear differential operator in one variable x . We assume that the $a_j(x)$ are holomorphic on a neighborhood of $x = 0$. Then we say that the origin 0 is a regular singularity of $Pu = 0$ if

$$(*) \quad \text{ord}_{x=0} a_j(x) \geq \text{ord}_{x=0} a_m(x) - (m - j).$$

Here $\text{ord}_{x=0}$ means the order of the zero. In this case, the local structure of the equation is very simple. In fact, the \mathcal{D}_X -module $\mathcal{D}_X/\mathcal{D}_X P$ is a direct sum of copies of the following modules:

$$\begin{aligned} \mathcal{O}_X &= \mathcal{D}_X/\mathcal{D}_X \partial, \mathcal{B}_{\{0\}|X} = \mathcal{D}_X/\mathcal{D}_X x, \mathcal{D}_X/\mathcal{D}_X (x\partial - \lambda)^{m+1} \quad (\lambda \in \mathbb{C}, m \in \mathbb{N}), \\ &\mathcal{D}_X/\mathcal{D}_X (x\partial)^{m+1} x \quad (m \in \mathbb{N}), \mathcal{D}_X/\mathcal{D}_X \partial (x\partial)^{m+1} \quad (m \in \mathbb{N}). \end{aligned}$$

If we denote by u the canonical generator, then we have $Pu = 0$. By multiplying either a power of ∂ or a power of x , we obtain

$$\sum_{j=0}^N b_j(x) (x\partial)^j u = 0$$

with $b_N(x) = 1$. Hence $\mathcal{F} = \sum_{j=0}^{\infty} \mathcal{O}(x\partial)^j u = \sum_{j=0}^{N-1} \mathcal{O}(x\partial)^j u$ is a coherent \mathcal{O} -submodule of \mathcal{M} which satisfies $(x\partial)\mathcal{F} \subset \mathcal{F}$. We shall generalize this property to the case of several variables.

9.2. Let X be a complex manifold, Ω an open subset of T^*X and V a closed involutive complex submanifold of Ω . Let us define

$$\mathcal{I}_V = \{u \in \mathcal{E}(1)|_\Omega; \sigma_1(P)|_V = 0\}$$

and let \mathcal{E}_V be the subring of $\mathcal{E}_X|_\Omega$ generated by \mathcal{I}_V . For a coherent \mathcal{E}_X -module \mathcal{M} , a coherent sub- $\mathcal{E}_X(0)$ -module \mathcal{L} of \mathcal{M} is called a *lattice* of \mathcal{M} if $\mathcal{M} = \mathcal{E}_X \mathcal{L}$. The following proposition is easily derived from the fact that $\mathcal{E}(0)$ is a Noetherian ring.

PROPOSITION 9.2.1 ([K-O] Theorem 1.4.7). *Let \mathcal{M} be a coherent $\mathcal{E}_X|_\Omega$ -module. Then the following conditions are equivalent.*

- (1) *For any point $p \in \Omega$, there is a lattice \mathcal{M}_0 of \mathcal{M} on a neighborhood of p such that $\mathcal{I}_V \mathcal{M}_0 = \mathcal{M}_0$.*
- (2) *For any open subset U of Ω and for any coherent $\mathcal{E}(0)$ -submodule \mathcal{L} of $\mathcal{M}|_U$, $\mathcal{E}_V \mathcal{L}$ is coherent over $\mathcal{E}(0)|_U$.*

Definition 9.2.2. If the equivalent conditions of the preceding proposition are satisfied, then we say that \mathcal{M} has *regular singularities* along V .

Remark that if \mathcal{M} has regular singularity along V , then the support of \mathcal{M} is contained in V . Let us denote by $IR_V(\mathcal{M})$ the set of points p such that \mathcal{M} has no regular singularities along V on any neighborhood of p .

The following theorem is an immediate consequence of Gabber's Theorem 6.3.2.

THEOREM 9.2.3. *$IR_V(\mathcal{M})$ is an involutive analytic subset of \mathcal{M} .*

In fact, if we take a lattice \mathcal{M}_0 of \mathcal{M} , then $T^*X \setminus IR_V(\mathcal{M})$ is the largest open subset on which $\mathcal{E}_V \mathcal{M}_0$ is coherent over $\mathcal{E}(0)$.

9.3. If an \mathcal{E} -module \mathcal{M} has regular singularities along an involutive submanifold V then \mathcal{M} is, roughly speaking, constant along the bicharacteristics of V . More precisely, let Y and Z be complex manifolds and $X = Y \times Z$. Let $z_0 \in Z$ and let j be the inclusion map $Y \hookrightarrow X$ by $y \mapsto (y, z_0)$. Then we have

THEOREM 9.3.1. *Let \mathcal{M} be a coherent \mathcal{E}_X -module. Assume that \mathcal{M} has regular singularities along $T^*Y \times T^*_Z Z$. Then \mathcal{M} is isomorphic to $j^* \mathcal{M} \hat{\otimes} \mathcal{O}_Z$.*

Note that any involutive submanifold V of T^*X with $\theta_X|_V \neq 0$ is transformed by a homogeneous symplectic transformation to the form $T^*Y \times T^*_Z Z$.

9.4. Noting that any nowhere dense closed analytic subset of a Lagrangean variety is never involutive, Theorem 9.2.3 implies the following theorem.

THEOREM 9.4.1. *Let \mathcal{M} be a holonomic \mathcal{E}_X -module. Then the following conditions are equivalent.*

- (i) *There exists a Lagrangean subvariety Λ such that \mathcal{M} has regular singularities along Λ .*
- (ii) *For any involutive subvariety Λ which contains $\text{Supp } \mathcal{M}$, \mathcal{M} has regular singularities along Λ .*
- (iii) *There exists an open dense subset Ω of $\text{Supp } \mathcal{M}$ such that \mathcal{M} has regular singularities along $\text{Supp } \mathcal{M}$ on Ω .*

If these equivalent conditions are satisfied, we say that \mathcal{M} is a *regular holonomic \mathcal{E}_X -module*.

The following properties are almost immediate.

THEOREM 9.4.2.

- (i) *Let $0 \rightarrow \mathcal{M}' \rightarrow \mathcal{M} \rightarrow \mathcal{M}'' \rightarrow 0$ be an exact sequence of three coherent \mathcal{E}_X -modules. If two of them are regular holonomic then so is the third.*
- (ii) *If \mathcal{M} is regular holonomic, its dual \mathcal{M}^* is also regular holonomic.*

We just mention another analytic property of regular holonomic modules, which generalizes the fact that a formal solution of an ordinary differential equation with regular singularity converges.

THEOREM 9.4.3 ([KK] Theorem 6.1.3). *If \mathcal{M} and \mathcal{N} are regular holonomic \mathcal{E}_X -modules, then $\mathcal{E}xt_{\mathcal{E}_X}^j(\mathcal{M}, \mathcal{N}) \rightarrow \mathcal{E}xt_{\mathcal{E}_X}^j(\mathcal{M}, \widehat{\mathcal{E}}_X \otimes_{\mathcal{E}_X} \mathcal{N})$ and $\mathcal{E}xt_{\mathcal{E}_X}^j(\mathcal{M}, \mathcal{N}) \rightarrow \mathcal{E}xt_{\mathcal{E}_X}^j(\mathcal{M}, \mathcal{E}_X^\infty \otimes_{\mathcal{E}_X} \mathcal{N})$ are isomorphisms.*

§ 10. STRUCTURE OF REGULAR HOLONOMIC \mathcal{E} -MODULES

(See [SKK], [KK])

10.1. Let Λ be a Lagrangean submanifold of T^*X . We define \mathcal{I}_Λ and \mathcal{E}_Λ as in § 9.2.

Then $\mathcal{E}_\Lambda(-1) = \mathcal{E}_\Lambda \cdot \mathcal{E}(-1)$ is a two-sided ideal of \mathcal{E}_Λ and $\mathcal{E}_\Lambda/\mathcal{E}_\Lambda(-1)$ is a sheaf of rings which contains $\mathcal{O}_\Lambda(0) = \mathcal{E}(0)/\mathcal{I}_\Lambda(-1)$, the sheaf of homogeneous functions on Λ .