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$$\partial_x \mapsto \alpha \partial_x - \beta x \partial_t$$

$$x \mapsto \gamma \partial_x \partial_t^{-1} + \delta x$$

$$\partial_t \mapsto \partial_t$$

$$\begin{aligned} t \mapsto t + \frac{1}{2} \{ & \langle \partial_x, {}^t \gamma \alpha \partial_x \rangle \partial_t^{-2} + \langle \partial_x, {}^t \gamma \beta x \rangle \partial_t^{-1} \\ & + \langle {}^t \gamma \beta x, \partial_x \rangle \partial_t^{-1} + \langle x, {}^t \delta \beta x \rangle \}. \end{aligned}$$

Then we have  $\Psi_{g_1} \Psi_{g_2} = \Psi_{g_1 g_2}$ .

## § 8. FUNCTORIAL PROPERTIES OF MICRO-DIFFERENTIAL MODULES (See [SKK])

### 8.1. External Tensor Product.

Let  $X$  and  $Y$  be complex manifolds and let  $p_1$  and  $p_2$  be the projections  $T^*(X \times Y) \rightarrow T^*X$  and  $T^*(X \times Y) \rightarrow T^*Y$ , respectively. Then  $\mathcal{E}_{X \times Y}$  contains  $p_1^{-1}\mathcal{E}_X \otimes_{\mathbb{C}} p_2^{-1}\mathcal{E}_Y$  as a subring. For an  $\mathcal{E}_X$ -module  $\mathcal{M}$  and an  $\mathcal{E}_Y$ -module  $\mathcal{N}$ , we define the  $\mathcal{E}_{X \times Y}$ -module  $\mathcal{M} \hat{\otimes} \mathcal{N}$  by

$$(8.1.1) \quad \mathcal{M} \hat{\otimes} \mathcal{N} = \mathcal{E}_{X \times Y} \underset{\substack{p_1^{-1}\mathcal{E}_X \otimes_{\mathbb{C}} p_2^{-1}\mathcal{E}_Y}}{\otimes} (p_1^{-1}\mathcal{M} \underset{\mathbb{C}}{\otimes} p_2^{-1}\mathcal{N}).$$

Then one can easily see

#### PROPOSITION 8.1.1.

- (i)  $\mathcal{M} \hat{\otimes} \mathcal{N}$  is an exact functor in  $\mathcal{M}$  and in  $\mathcal{N}$  and  $\text{Supp}(\mathcal{M} \hat{\otimes} \mathcal{N}) = \text{Supp } \mathcal{M} \times \text{Supp } \mathcal{N}$ .
- (ii) If  $\mathcal{M}$  is  $\mathcal{E}_X$ -coherent and  $\mathcal{N}$  is  $\mathcal{E}_Y$ -coherent, then  $\mathcal{M} \hat{\otimes} \mathcal{N}$  is  $\mathcal{E}_{X \times Y}$ -coherent.

8.2. For a complex submanifold  $Y$  of a complex manifold  $X$  of codimension  $l$ , the sheaf  $\lim_{\substack{\rightarrow \\ m}} \mathcal{E}xt_{\mathcal{O}_X}^l(\mathcal{O}_X/\mathcal{J}^m, \mathcal{O}_X)$  has a natural structure of  $\mathcal{D}_X$ -module,

which is denoted by  $\mathcal{B}_{Y|X}$ . Here  $\mathcal{J}$  is the defining ideal of  $Y$ . The homomorphism  $\mathcal{O}_Y \rightarrow \mathcal{E}xt_{\mathcal{O}_X}^l(\mathcal{O}_Y, \Omega_X^l) \rightarrow \Omega_X^l \otimes_{\mathcal{O}_X} \mathcal{B}_{Y|X}$  gives the canonical section  $c(Y, X)$  of  $\Omega_X^l \otimes_{\mathcal{O}_X} \mathcal{B}_{Y|X}$ . If we take local coordinates  $(x_1, \dots, x_n)$  of  $X$  such that  $Y$  is defined by  $x_1 = \dots = x_l = 0$ , then we have

$$\mathcal{B}_{Y|X} \cong \mathcal{D}_X / \sum_{j \leq l} \mathcal{D}_X x_j + \sum_{j > l} \mathcal{D}_X \partial_j.$$

If we denote by  $\delta$  the canonical generator of the left hand side, then  $c(Y, X)$  corresponds to  $dx_1 \wedge \dots \wedge dx_l \otimes \delta$ . We set

$$\mathcal{C}_{Y|X} = \mathcal{E}_X \underset{\pi^{-1}\mathcal{D}_X}{\otimes} \pi^{-1}\mathcal{B}_{Y|X}.$$

Therefore locally we have

$$\mathcal{C}_{Y|X} \cong \mathcal{E}_X / \sum_{j \leq d} \mathcal{E}_X x_j + \sum_{j > d} \mathcal{E}_X \partial_j.$$

Then  $\mathcal{C}_{Y|X}$  is a coherent  $\mathcal{E}_X$ -module whose support is  $T_Y^*X$ .

8.3. For an invertible  $\mathcal{O}_X$ -module  $\mathcal{L}$ ,  $\mathcal{L} \underset{\mathcal{O}_X}{\otimes} \mathcal{E}_X \underset{\mathcal{O}_X}{\otimes} \mathcal{L}^{\otimes -1}$  has a natural structure of sheaves of rings, by the composition rule

$$(s \otimes P \otimes s^{\otimes -1}) \circ (s \otimes Q \otimes s^{\otimes -1}) = s \otimes PQ \otimes s^{\otimes -1}$$

for an invertible section  $s$  of  $\mathcal{L}$  and  $P, Q \in \mathcal{E}_X$ .

Then the category  $\text{Mod}(\mathcal{E}_X)$  of left  $\mathcal{E}_X$ -modules and the category  $\text{Mod}(\mathcal{L} \underset{\mathcal{O}_X}{\otimes} \mathcal{E}_X \underset{\mathcal{O}_X}{\otimes} \mathcal{L}^{\otimes -1})$  of left  $(\mathcal{L} \underset{\mathcal{O}_X}{\otimes} \mathcal{E}_X \underset{\mathcal{O}_X}{\otimes} \mathcal{L}^{\otimes -1})$ -modules are equivalent by the functor

$$\text{Mod}(\mathcal{E}_X) \ni \mathcal{M} \mapsto \mathcal{L} \underset{\mathcal{O}_X}{\otimes} \mathcal{M} \in \text{Mod}(\mathcal{L} \underset{\mathcal{O}_X}{\otimes} \mathcal{E}_X \underset{\mathcal{O}_X}{\otimes} \mathcal{L}^{\otimes -1}).$$

8.4. Let  $\omega_X$  be the canonical sheaf on  $X$ , i.e. the sheaf of differential forms with top degree. Let  $a$  be the antipodal map of  $T^*X$ , i.e. the multiplication by  $-1$ . Then we have the anti-ring isomorphism.

$$(8.4.1) \quad \omega_X \underset{\mathcal{O}_X}{\otimes} \mathcal{E}_X \underset{\mathcal{O}_X}{\otimes} \omega_X^{\otimes -1} \xrightarrow{\sim} a^{-1} \mathcal{E}_X.$$

This homomorphism is given by using a local coordinate system  $(x_1, \dots, x_n)$  as follows. For  $P = \sum P_j(x, \partial) \in \mathcal{E}_X$  we define  $P^* = \sum P_j^*(x, \partial)$ , called the formal adjoint of  $P$  ([SKK] Chap. II, Th. 1.5.1), by

$$(8.4.2) \quad P_j^*(x, -\xi) = \sum_{\substack{j=l-|\alpha| \\ \alpha \in \mathbb{N}^n}} \frac{(-1)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha \partial_x^\alpha P_j(x, \xi).$$

This is well-defined and satisfies

$$(8.4.3) \quad (P^*)^* = P$$

$$(8.4.4) \quad (PQ)^* = Q^*P^*.$$

Then the isomorphism (8.4.1) is given by

$$(8.4.5) \quad dx \otimes P \otimes (dx)^{\otimes -1} \mapsto P^*$$

where  $dx = dx_1 \wedge \dots \wedge dx_n \in \omega_X$ . This is independent of coordinate transformations.

8.5. The isomorphism (8.4.1) can be explained as follows. Let  $\Delta_X$  be the diagonal set of  $X \times X$ , and let  $p_j$  be the  $j$ -th projection from  $T_{\Delta_X}^*(X \times X)$  to  $T^*X$  for  $j = 1, 2$ . Then the  $p_j$  are isomorphisms and  $p_2 \circ p_1^{-1} = a$ . Let  $q_j$  be the  $j$ -th projection from  $T^*(X \times X)$  to  $X$  ( $j = 1, 2$ ). Then  $c(\Delta_X, X \times X)$  gives the canonical section of  $q_2^{-1}\omega_X \otimes_{q_2^{-1}\mathcal{O}_X} \mathcal{C}_{\Delta_X|X \times X}$ . Since  $\mathcal{C}_{\Delta_X|X \times X}$  is a  $p_1^{-1}\mathcal{E}_X$ -module, this section gives a homomorphism

$$p_1^{-1}\mathcal{E}_X \rightarrow q_2^{-1}\omega_X \otimes_{q_2^{-1}\mathcal{O}_X} \mathcal{C}_{\Delta_X|X \times X}.$$

It turns out that this is an isomorphism and the right multiplication of  $\mathcal{O}_X$  on  $\mathcal{E}_X$  corresponds to the  $\mathcal{O}_X$ -module structure of  $q_2^{-1}\omega_X \otimes_{q_2^{-1}\mathcal{O}_X} \mathcal{C}_{\Delta_X|X \times X}$  via  $q_2$ . Thus we obtain

$$p_1^{-1}(\omega_X \otimes_{\mathcal{O}_X} \mathcal{E}_X \otimes_{\mathcal{O}_X} \omega_X^{\otimes -1}) \xrightarrow{\sim} q_1^{-1}\omega_X \otimes_{q_1^{-1}\mathcal{O}_X} \mathcal{C}_{\Delta_X|X \times X}.$$

This last being isomorphic to  $p_2^{-1}\mathcal{E}_X$ , we obtain

$$\omega_X \otimes_{\mathcal{O}_X} \mathcal{E}_X \otimes_{\mathcal{O}_X} \omega_X^{\otimes -1} \xrightarrow{\sim} p_1 p_2^{-1} \mathcal{E}_X \simeq a^{-1} \mathcal{E}_X.$$

8.6. By 8.3 and 8.4, if  $\mathcal{M}$  is a left  $\mathcal{E}_{X|U}$ -module for an open set  $U$  of  $T^*X$ , then  $\omega_X \otimes_{\mathcal{O}_X} a^{-1} \mathcal{M}$  is a right  $(\mathcal{E}_{X|aU})$ -module.

8.7. For a left coherent  $\mathcal{E}_X$ -module  $\mathcal{M}$ ,  $\mathcal{E}xt_{\mathcal{E}_X}^j(\mathcal{M}, \mathcal{E}_X)$  is a right coherent  $\mathcal{E}_X$ -module. Therefore  $\mathcal{E}xt_{\mathcal{E}_X}^j(\mathcal{M}, \mathcal{E}_X) \otimes_{\mathcal{O}_X} \omega_X^{\otimes -1}$  is a left  $\mathcal{E}_X$ -module by § 8.6. If  $\mathcal{M}$  is holonomic then  $\mathcal{E}xt_{\mathcal{E}_X}^j(\mathcal{M}, \mathcal{E}_X) = 0$  for  $j \neq n = \dim X$  (See [SKK], [K1]). Set  $\mathcal{M}^* = \mathcal{E}xt_{\mathcal{E}_X}^n(\mathcal{M}, \mathcal{E}_X) \otimes_{\mathcal{O}_X} \omega_X^{\otimes -1}$ . Then  $\mathcal{M}^*$  is also a holonomic  $\mathcal{E}_X$ -module.

We call  $\mathcal{M}^*$  the *dual system* of  $\mathcal{M}$ . We have  $\mathcal{M}^{**} = \mathcal{M}$ , and  $\mathcal{M} \mapsto \mathcal{M}^*$  is an exact contravariant functor on the category of holonomic  $\mathcal{E}_X$ -modules.

8.8. Let  $X$  and  $Y$  be complex manifolds, and let  $p_1: T^*(X \times Y) \rightarrow T^*X$  and  $p_2: T^*(X \times Y) \rightarrow T^*Y$  be the canonical projections. Let  $p_2^a$  denote  $p_2 \circ a$ . Let  $\mathcal{K}$  be a left  $\mathcal{E}_{X \times Y}$ -module defined on an open subset  $\Omega$  of  $T^*(X \times Y)$ . Then, by § 8.6,  $\omega_Y \otimes_{\mathcal{O}_Y} \mathcal{K}$  has a structure of  $(p_1^{-1}\mathcal{E}_X, p_2^{a-1}\mathcal{E}_Y)$ -bi-module. For an  $\mathcal{E}_Y$ -module  $\mathcal{N}$ ,

$$\mathcal{M} = p_{1*}((\omega_Y \otimes_{\mathcal{O}_Y} \mathcal{K}) \otimes_{p_2^{a-1}\mathcal{E}_Y} p_2^{a-1}\mathcal{N})$$

has a structure of  $\mathcal{E}_X$ -module. We have the following

**THEOREM 8.8.1.** *Let  $\Omega$ ,  $U_X$  and  $U_Y$  be open subsets of  $T^*(X \times Y)$ ,  $T^*X$  and  $T^*Y$ , respectively. Let  $\mathcal{K}$  be a coherent  $(\mathcal{E}_{X \times Y}|_\Omega)$ -module and  $\mathcal{N}$  a coherent  $(\mathcal{E}_Y|_{U_Y})$ -module. Assume*

(i)  $p_1: p_1^{-1}U_X \cap \text{Supp } \mathcal{K} \cap p_2^{a-1} \text{Supp } \mathcal{N} \rightarrow U_X$  is a finite morphism.

*Then we have*

(a)  $\mathcal{T}or_j^{p_2^{a-1}\mathcal{E}_Y}(\omega_Y \otimes_{\mathcal{O}_Y} \mathcal{K}, p_2^{a-1}\mathcal{N})|_{p_1^{-1}U_X} = 0$  for  $j \neq 0$ .

(b)  $\mathcal{M} = p_{1*}((\omega_Y \otimes_{\mathcal{O}_Y} \mathcal{K}) \otimes_{p_2^{a-1}\mathcal{E}_Y} p_2^{a-1}\mathcal{N})|_{U_X}$  is a coherent  $\mathcal{E}_X$ -module.

(c)  $\text{Supp } \mathcal{M} = U_X \cap p_1(\text{Supp } \mathcal{K} \cap p_2^{a-1} \text{Supp } \mathcal{N})$ .

We denote  $p_{1*}((\omega_Y \otimes_{\mathcal{O}_Y} \mathcal{K}) \otimes_{p_2^{a-1}\mathcal{E}_Y} p_2^{a-1}\mathcal{N})$  by  $\int_Y \mathcal{K} \circ \mathcal{N}$ .

8.9. Let  $f: X \rightarrow Y$  be a holomorphic map and let  $\Delta_f$  be the graph of  $f$ , i.e.  $\{(x, f(x)) \in X \times Y; x \in X\}$ , then  $\mathcal{K} = \mathcal{C}_{\Delta_f|X \times Y}$  is a coherent  $\mathcal{E}_{X \times Y}$ -module whose support is  $T_{\Delta_f}^*(X \times Y)$ . Now let  $\tilde{\omega}$  be the canonical map  $X \times_{Y} T^*Y \rightarrow T^*X$  and  $\rho$  the projection  $X \times_{Y} T^*Y \rightarrow T^*Y$ . Then we have the following diagram

$$(8.9.1) \quad \begin{array}{ccccc} T^*X & \xleftarrow{\tilde{\omega}_f} & X \times_{Y} T^*Y & \xrightarrow{\rho_f} & T^*Y \\ id \parallel & & \downarrow & & \parallel id \\ T^*X & \xleftarrow{p_1} & T_{\Delta_f}^*(X \times Y) & \xrightarrow{p_2} & T^*Y \end{array}$$

We set  $\mathcal{E}_{X \rightarrow Y} = \omega_Y \otimes_{\mathcal{O}_Y} \mathcal{C}_{\Delta_f|X \times Y}$  and consider this as a sheaf on  $X \times_{T^*Y} T^*Y$  by the above isomorphism. Then  $\mathcal{E}_{X \rightarrow Y}$  is a  $(\tilde{\omega}^{-1}\mathcal{E}_X, \rho^{-1}\mathcal{E}_Y)$ -bi-module. For an  $\mathcal{E}_Y$ -module  $\mathcal{N}$ ,

$$\int \mathcal{K} \circ \mathcal{N} = \mathbf{R}\tilde{\omega}_* \rho^{-1}(\mathcal{E}_{X \rightarrow Y} \otimes_{\rho^{-1}\mathcal{E}_Y} \rho^{-1}\mathcal{N}).$$

We shall denote this by  $f^*\mathcal{N}$  and call it the pull-back of  $\mathcal{N}$ . Then Theorem 8.8.1 reads as follows.

**THEOREM 8.9.1.** *Let  $U_X$  and  $U_Y$  be open subsets of  $T^*X$  and  $T^*Y$ , respectively. Let  $\mathcal{N}$  be a coherent  $(\mathcal{E}_Y|_U)$ -module. Assume*

- (i)  $\rho_f^{-1}(\text{Supp } \mathcal{N}) \cap \tilde{\omega}_f^{-1}(U_X) \rightarrow U_X$  is a finite morphism.

*Then we have*

(a)  $\mathcal{T}or_j^{\rho_f^{-1}\mathcal{E}_Y}(\mathcal{E}_{X \rightarrow Y}, \mathcal{N}) = 0$  for  $j \neq 0$ .

(b)  $\mathcal{M} = \tilde{\omega}_f_*(\mathcal{E}_{X \rightarrow Y} \otimes_{\rho_f^{-1}\mathcal{E}_Y} \rho_f^{-1}\mathcal{N})|_{U_X}$  is a coherent  $\mathcal{E}_X$ -module.

(c)  $\text{Supp } M = \tilde{\omega}_f \rho_f^{-1} \text{Supp } \mathcal{N} \cap U_X$ .

8.10. Similarly let  $g: Y \rightarrow X$  be a holomorphic map and let  $\Delta_g$  be the graph of  $g$ , i.e.  $\{(g(y), y) \in X \times Y; y \in Y\}$ . Then we have the isomorphisms

$$(8.10.1) \quad \begin{array}{ccccc} T^*X & \xleftarrow{\rho_g} & Y \times T^*X & \xrightarrow{\tilde{\omega}_g} & T^*Y \\ \parallel & & \parallel & & \parallel \text{ id.} \\ T^*X & \xleftarrow[p_1]{} & T^*_{\Delta_g}(X \times Y) & \xrightarrow[p_2]{} & T^*Y. \end{array}$$

We set  $\mathcal{E}_{X \leftarrow Y} = \omega_Y \otimes_{\mathcal{O}_Y} \mathcal{C}_{\Delta_g|X \times Y}$  and regard this as a sheaf on  $Y \times_{T^*X} T^*X$ . Then  $\mathcal{E}_{X \leftarrow Y}$  is a  $(\rho^{-1}\mathcal{E}_X, \tilde{\omega}^{-1}\mathcal{E}_Y)$ -bi-module. For an  $\mathcal{E}_Y$ -module  $\mathcal{N}$  we have

$$\int \mathcal{C}_{\Delta_g|X \times Y} \circ \mathcal{N} = \mathbf{R}\rho_* \tilde{\omega}^{-1}(\mathcal{E}_{X \leftarrow Y} \otimes_{\tilde{\omega}^{-1}\mathcal{E}_Y} \tilde{\omega}^{-1}\mathcal{N}).$$

We shall denote this by  $\int_g \mathcal{N}$ . Then Theorem 8.8.1 applies to this case and we have

**THEOREM 8.10.1.** *Let  $U_X$  and  $U_Y$  be open subsets of  $T^*X$  and  $T^*Y$ , respectively. Let  $\mathcal{N}$  be a coherent  $(\mathcal{E}_Y|_{U_Y})$ -module. Assume*

- (i)  $\rho_g: \tilde{\omega}_g^{-1}(\text{Supp } \mathcal{N}) \cap \rho_g^{-1}(U_X) \rightarrow U_X$  is a finite morphism.

*Then we have*

- (a)  $\mathcal{T}or_j^{\tilde{\omega}_g^{-1}\mathcal{E}_Y}(\mathcal{E}_{X \leftarrow Y}, \tilde{\omega}_g^{-1}\mathcal{N}) = 0$  for  $j \neq 0$ .
- (b)  $\mathcal{M} = \rho_{g*}(\mathcal{E}_{X \leftarrow Y} \otimes_{\tilde{\omega}_g^{-1}\mathcal{E}_Y} \tilde{\omega}_g^{-1}\mathcal{N})|_{U_X}$  is a coherent  $\mathcal{E}_X|_{U_X}$ -module.
- (c)  $\text{Supp } \mathcal{M} = \rho_g(\tilde{\omega}_g^{-1} \text{Supp } \mathcal{N} \cap U_X)$ .

## § 9. REGULARITY CONDITIONS (See [KK], [K-O])

9.1. Let us recall the notion of regular singularity of ordinary differential equations. Let  $P(x, \partial) = \sum_{j \leq m} a_j(x)\partial^j$  be a linear differential operator in one variable  $x$ . We assume that the  $a_j(x)$  are holomorphic on a neighborhood of  $x = 0$ . Then we say that the origin 0 is a regular singularity of  $Pu = 0$  if

$$(*) \quad \text{ord}_{x=0} a_j(x) \geq \text{ord}_{x=0} a_m(x) - (m-j).$$

Here  $\text{ord}_{x=0}$  means the order of the zero. In this case, the local structure of the equation is very simple. In fact, the  $\mathcal{D}_x$ -module  $\mathcal{D}_x/\mathcal{D}_x P$  is a direct sum of copies of the following modules:

$$\begin{aligned} \mathcal{O}_x &= \mathcal{D}_x/\mathcal{D}_x \partial, \quad \mathcal{B}_{\{0\}|X} = \mathcal{D}_x/\mathcal{D}_x x, \quad \mathcal{D}_x/\mathcal{D}_x(x\partial - \lambda)^{m+1} \quad (\lambda \in \mathbb{C}, m \in \mathbb{N}), \\ &\quad \mathcal{D}_x/\mathcal{D}_x(x\partial)^{m+1}x \quad (m \in \mathbb{N}), \quad \mathcal{D}_x/\mathcal{D}_x \partial(x\partial)^{m+1} \quad (m \in \mathbb{N}). \end{aligned}$$

If we denote by  $u$  the canonical generator, then we have  $Pu = 0$ . By multiplying either a power of  $\partial$  or a power of  $x$ , we obtain

$$\sum_{j=0}^N b_j(x)(x\partial)^j u = 0$$

with  $b_N(x) = 1$ . Hence  $\mathcal{F} = \sum_{j=0}^{\infty} \mathcal{O}(x\partial)^j u = \sum_{j=0}^{N-1} \mathcal{O}(x\partial)^j u$  is a coherent  $\mathcal{O}$ -submodule of  $\mathcal{M}$  which satisfies  $(x\partial)\mathcal{F} \subset \mathcal{F}$ . We shall generalize this property to the case of several variables.

9.2. Let  $X$  be a complex manifold,  $\Omega$  an open subset of  $\overset{\circ}{T^*X}$  and  $V$  a closed involutive complex submanifold of  $\Omega$ . Let us define