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$$\partial_x \mapsto \alpha\partial_x - \beta x \partial_t$$
$$x \mapsto \gamma\partial_x\partial_t^{-1} + \delta x$$
$$\partial_t \mapsto \partial_t$$
$$t \mapsto t + \frac{1}{2}\{<\partial_x, {}^t\gamma\alpha\partial_x> \partial_t^{-2} + <\partial_x, {}^t\gamma\beta x> \partial_t^{-1}$$
$$+ <{}^t\gamma\beta x, \partial_x> \partial_t^{-1} + <x, {}^t\delta\beta x>\}.$$

Then we have $\Psi_{g_1}\Psi_{g_2} = \Psi_{g_1g_2}$.

§ 8. Functorial Properties of Micro-Differential Modules (See [SKK])

8.1. External Tensor Product.

Let X and Y be complex manifolds and let p_1 and p_2 be the projections $T^*(X \times Y) \to T^*X$ and $T^*(X \times Y) \to T^*Y$, respectively. Then $\mathscr{E}_{X \times Y}$ contains $p_1^{-1}\mathscr{E}_X \underset{\mathbf{C}}{\otimes} p_2^{-1}\mathscr{E}_Y$ as a subring. For an $\mathscr{E}_X$-module $\mathscr{M}$ and an $\mathscr{E}_Y$-module $\mathscr{N}$, we define the $\mathscr{E}_{X\times Y}$-module $\mathscr{M} \hat{\otimes} \mathscr{N}$ by

(8.1.1) $$\mathscr{M} \hat{\otimes} \mathscr{N} = \mathscr{E}_{X\times Y} \underset{p_1^{-1}\mathscr{E}_X \underset{\mathbf{C}}{\otimes} p_2^{-1}\mathscr{E}_Y}{\otimes} (p_1^{-1}\mathscr{M} \underset{\mathbf{C}}{\otimes} p_2^{-1}\mathscr{N}).$$

Then one can easily see

Proposition 8.1.1.

(i) $\mathscr{M} \hat{\otimes} \mathscr{N}$ *is an exact functor in* $\mathscr{M}$ *and in* $\mathscr{N}$ *and* $\operatorname{Supp}(\mathscr{M} \hat{\otimes} \mathscr{N}) = \operatorname{Supp}\mathscr{M} \times \operatorname{Supp}\mathscr{N}$.

(ii) *If* $\mathscr{M}$ *is* $\mathscr{E}_X$*-coherent and* $\mathscr{N}$ *is* $\mathscr{E}_Y$*-coherent, then* $\mathscr{M} \hat{\otimes} \mathscr{N}$ *is* $\mathscr{E}_{X\times Y}$*-coherent.*

8.2. For a complex submanifold Y of a complex manifold X of codimension l, the sheaf $\varinjlim_m \mathscr{E}xt^l_{\mathcal{O}_X}(\mathcal{O}_X/\mathscr{J}^m, \mathcal{O}_X)$ has a natural structure of $\mathscr{D}_X$-module, which is denoted by $\mathscr{B}_{Y|X}$. Here $\mathscr{J}$ is the defining ideal of Y. The homomorphism $\mathcal{O}_Y \to \mathscr{E}xt^l_{\mathcal{O}_X}(\mathcal{O}_Y, \Omega^l_X) \to \Omega^l_X \underset{\mathcal{O}_X}{\otimes} \mathscr{B}_{Y|X}$ gives the canonical section $c(Y, X)$ of $\Omega^l_X \underset{\mathcal{O}_X}{\otimes} \mathscr{B}_{Y|X}$. If we take local coordinates $(x_1, ..., x_n)$ of X such that Y is defined by $x_1 = ... = x_l = 0$, then we have

$$\mathscr{B}_{Y|X} \cong \mathscr{D}_X / \sum_{j \leqslant l} \mathscr{D}_X x_j + \sum_{j > l} \mathscr{D}_X \partial_j .$$

If we denote by δ the canonical generator of the left hand side, then $c(Y, X)$ corresponds to $dx_1 \wedge \dots \wedge dx_l \otimes \delta$. We set

$$\mathscr{C}_{Y|X} = \mathscr{E}_X \underset{\pi^{-1}\mathscr{D}_X}{\otimes} \pi^{-1}\mathscr{B}_{Y|X} .$$

Therefore locally we have

$$\mathscr{C}_{Y|X} \cong \mathscr{E}_X / \sum_{j \leqslant d} \mathscr{E}_X x_j + \sum_{j > d} \mathscr{E}_X \partial_j .$$

Then $\mathscr{C}_{Y|X}$ is a coherent $\mathscr{E}_X$-module whose support is T_Y^*X.

8.3. For an invertible $\mathcal{O}_X$-module $\mathscr{L}$, $\mathscr{L} \underset{\mathcal{O}_X}{\otimes} \mathscr{E}_X \underset{\mathcal{O}_X}{\otimes} \mathscr{L}^{\otimes -1}$ has a natural structure of sheaves of rings, by the composition rule

$$(s \otimes P \otimes s^{\otimes -1}) \circ (s \otimes Q \otimes s^{\otimes -1}) = s \otimes PQ \otimes s^{\otimes -1}$$

for an invertible section s of $\mathscr{L}$ and $P, Q \in \mathscr{E}_X$.

Then the category $\mathrm{Mod}(\mathscr{E}_X)$ of left $\mathscr{E}_X$-modules and the category $\mathrm{Mod}(\mathscr{L} \underset{\mathcal{O}_X}{\otimes} \mathscr{E}_X \underset{\mathcal{O}_X}{\otimes} \mathscr{L}^{\otimes -1})$ of left $(\mathscr{L} \underset{\mathcal{O}_X}{\otimes} \mathscr{E}_X \underset{\mathcal{O}_X}{\otimes} \mathscr{L}^{\otimes -1})$-modules are equivalent by the functor

$$\mathrm{Mod}(\mathscr{E}_X) \ni \mathscr{M} \mapsto \mathscr{L} \underset{\mathcal{O}_X}{\otimes} \mathscr{M} \in \mathrm{Mod}(\mathscr{L} \underset{\mathcal{O}_X}{\otimes} \mathscr{E}_X \underset{\mathcal{O}_X}{\otimes} \mathscr{L}^{\otimes -1}) .$$

8.4. Let ω_X be the canonical sheaf on X, i.e. the sheaf of differential forms with top degree. Let a be the antipodal map of T^*X, i.e. the multiplication by -1. Then we have the anti-ring isomorphism.

$$\omega_X \underset{\mathcal{O}_X}{\otimes} \mathscr{E}_X \underset{\mathcal{O}_X}{\otimes} \omega_X^{\otimes -1} \xrightarrow{\sim} a^{-1}\mathscr{E}_X . \tag{8.4.1}$$

This homomorphism is given by using a local coordinate system $(x_1, \dots, x_n)$ as follows. For $P = \sum P_j(x, \partial) \in \mathscr{E}_X$ we define $P^* = \sum P_j^*(x, \partial)$, called the formal adjoint of P ([SKK] Chap. II, Th. 1.5.1), by

$$P_l^*(x, -\xi) = \sum_{\substack{j = l - |\alpha| \\ \alpha \in \mathbf{N}^n}} \frac{(-1)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha \partial_x^\alpha P_j(x, \xi) . \tag{8.4.2}$$

This is well-defined and satisfies

$$(P^*)^* = P \tag{8.4.3}$$

(8.4.4) $$(PQ)^* = Q^*P^* .$$

Then the isomorphism (8.4.1) is given by

(8.4.5) $$dx \otimes P \otimes (dx)^{\otimes -1} \mapsto P^*$$

where $dx = dx_1 \wedge \ldots \wedge dx_n \in \omega_X$. This is independent of coordinate transformations.

8.5. The isomorphism (8.4.1) can be explained as follows. Let Δ_X be the diagonal set of $X \times X$, and let p_j be the j-th projection from $T^*_{\Delta_X}(X \times X)$ to T^*X for $j = 1, 2$. Then the p_j are isomorphisms and $p_2 \circ p_1^{-1} = a$. Let q_j be the j-th projection from $T^*(X \times X)$ to $X (j=1, 2)$. Then $c(\Delta_X, X \times X)$ gives the canonical section of $q_2^{-1}\omega_X \underset{q_2^{-1}\mathcal{O}_X}{\otimes} \mathscr{C}_{\Delta_X|X\times X}$. Since $\mathscr{C}_{\Delta_X|X\times X}$ is a $p_1^{-1}\mathscr{E}_X$-module, this section gives a homomorphism

$$p_1^{-1}\mathscr{E}_X \to q_2^{-1}\omega_X \underset{q_2^{-1}\mathcal{O}_X}{\otimes} \mathscr{C}_{\Delta_X|X\times X} .$$

It turns out that this is an isomorphism and the right multiplication of $\mathcal{O}_X$ on $\mathscr{E}_X$ corresponds to the $\mathcal{O}_X$-module structure of $q_2^{-1}\omega_X \underset{q_2^{-1}\mathcal{O}_X}{\otimes} \mathscr{C}_{\Delta_X|X\times X}$ via q_2. Thus we obtain

$$p_1^{-1}(\omega_X \underset{\mathcal{O}_X}{\otimes} \mathscr{E}_X \underset{\mathcal{O}_X}{\otimes} \omega_X^{\otimes -1}) \xrightarrow{\sim} q_1^{-1}\omega_X \underset{q_1^{-1}\mathcal{O}_X}{\otimes} \mathscr{C}_{\Delta_X|X\times X} .$$

This last being isomorphic to $p_2^{-1}\mathscr{E}_X$, we obtain

$$\omega_X \underset{\mathcal{O}_X}{\otimes} \mathscr{E}_X \underset{\mathcal{O}_X}{\otimes} \omega_X^{\otimes -1} \xrightarrow{\sim} p_1 p_2^{-1}\mathscr{E}_X \simeq a^{-1}\mathscr{E}_X .$$

8.6. By 8.3 and 8.4, if $\mathscr{M}$ is a left $\mathscr{E}_{X|U}$-module for an open set U of T^*X, then $\omega_X \underset{\mathcal{O}_X}{\otimes} a^{-1}\mathscr{M}$ is a right $(\mathscr{E}_{X|aU})$-module.

8.7. For a left coherent $\mathscr{E}_X$-module $\mathscr{M}$, $\mathscr{E}xt^j_{\mathscr{E}_X}(\mathscr{M}, \mathscr{E}_X)$ is a right coherent $\mathscr{E}_X$-module. Therefore $\mathscr{E}xt^j_{\mathscr{E}_X}(\mathscr{M}, \mathscr{E}_X) \underset{\mathcal{O}_X}{\otimes} \omega_X^{\otimes -1}$ is a left $\mathscr{E}_X$-module by § 8.6. If $\mathscr{M}$ is holonomic then $\mathscr{E}xt^j_{\mathscr{E}_X}(\mathscr{M}, \mathscr{E}_X) = 0$ for $j \neq n = \dim X$ (See [SKK], [K1]). Set $\mathscr{M}^* = \mathscr{E}xt^n_{\mathscr{E}_X}(\mathscr{M}, \mathscr{E}_X) \underset{\mathcal{O}_X}{\otimes} \omega_X^{\otimes -1}$. Then $\mathscr{M}^*$ is also a holonomic $\mathscr{E}_X$-module.

We call $\mathscr{M}^*$ the *dual system* of $\mathscr{M}$. We have $\mathscr{M}^{**} = \mathscr{M}$, and $\mathscr{M} \mapsto \mathscr{M}^*$ is an exact contravariant functor on the category of holonomic $\mathscr{E}_X$-modules.

8.8. Let X and Y be complex manifolds, and let $p_1: T^*(X \times Y) \to T^*X$ and $p_2: T^*(X \times Y) \to T^*Y$ be the canonical projections. Let p_2^a denote $p_2 \circ a$. Let $\mathscr{K}$ be a left $\mathscr{E}_{X \times Y}$-module defined on an open subset Ω of $T^*(X \times Y)$. Then, by § 8.6, $\omega_Y \underset{\mathscr{O}_Y}{\otimes} \mathscr{K}$ has a structure of $(p_1^{-1}\mathscr{E}_X, p_2^{a-1}\mathscr{E}_Y)$-bi-module. For an $\mathscr{E}_Y$-module $\mathscr{N}$,

$$\mathscr{M} = p_{1*}\big((\omega_Y \underset{\mathscr{O}_Y}{\otimes} \mathscr{K}) \underset{p_2^{a-1}\mathscr{E}_Y}{\otimes} p_2^{a-1}\mathscr{N}\big)$$

has a structure of $\mathscr{E}_X$-module. We have the following

THEOREM 8.8.1. *Let Ω, U_X and U_Y be open subsets of $T^*(X \times Y)$, T^*X and T^*Y, respectively. Let $\mathscr{K}$ be a coherent $(\mathscr{E}_{X \times Y}|_\Omega)$-module and $\mathscr{N}$ a coherent $(\mathscr{E}_Y|_{U_Y})$-module. Assume*

(i) $p_1: p_1^{-1}U_X \cap \operatorname{Supp} \mathscr{K} \cap p_2^{a-1} \operatorname{Supp} \mathscr{N} \to U_X$ *is a finite morphism.*

Then we have

(a) $\mathscr{T}or_j^{p_2^{a-1}\mathscr{E}_Y}(\omega_Y \underset{\mathscr{O}_Y}{\otimes} \mathscr{K}, p_2^{a-1}\mathscr{N})|_{p_1^{-1}U_X} = 0$ *for* $j \neq 0$.

(b) $\mathscr{M} = p_{1*}\big((\omega_Y \underset{\mathscr{O}_Y}{\otimes} \mathscr{K}) \underset{p_2^{a-1}\mathscr{E}_Y}{\otimes} p_2^{a-1}\mathscr{N}\big)|_{U_X}$ is a coherent $\mathscr{E}_X$-module.

(c) $\operatorname{Supp} \mathscr{M} = U_X \cap p_1 (\operatorname{Supp} \mathscr{K} \cap p_2^{a-1} \operatorname{Supp} \mathscr{N})$.

We denote $p_{1*}\big((\omega_Y \underset{\mathscr{O}_Y}{\otimes} \mathscr{K}) \underset{p_2^{a-1}\mathscr{E}_Y}{\otimes} p_2^{a-1}\mathscr{N}\big)$ by $\int_Y \mathscr{K} \circ \mathscr{N}$.

8.9. Let $f: X \to Y$ be a holomorphic map and let Δ_f be the graph of f, i.e. $\{(x, f(x)) \in X \times Y; x \in X\}$, then $\mathscr{K} = \mathscr{C}_{\Delta_f|X \times Y}$ is a coherent $\mathscr{E}_{X \times Y}$-module whose support is $T^*_{\Delta_f}(X \times Y)$. Now let $\tilde{\omega}$ be the canonical map $X \underset{Y}{\times} T^*Y \to T^*X$ and ρ the projection $X \underset{Y}{\times} T^*Y$. Then we have the following diagram

$$\begin{array}{ccccc} T^*X & \overset{\tilde{\omega}_f}{\leftarrow} & X \underset{Y}{\times} T^*Y & \overset{\rho_f}{\rightarrow} & T^*Y \\ \text{id} \parallel & & \wr\!\mid & & \parallel \text{id}. \\ T^*X & \underset{p_1}{\leftarrow} & T^*_{\Delta_f}(X \times Y) & \underset{p_2^a}{\rightarrow} & T^*Y \end{array} \tag{8.9.1}$$

We set $\mathscr{E}_{X\to Y} = \omega_Y \underset{\mathscr{O}_Y}{\otimes} \mathscr{C}_{\Delta_f|X\times Y}$ and consider this as a sheaf on $X \underset{Y}{\times} T^*Y$ by the above isomorphism. Then $\mathscr{E}_{X\to Y}$ is a $(\tilde{\omega}^{-1}\mathscr{E}_X, \rho^{-1}\mathscr{E}_Y)$-bi-module. For an $\mathscr{E}_Y$-module $\mathscr{N}$,

$$\int \mathscr{K} \circ \mathscr{N} = \mathbf{R}\tilde{\omega}_*\rho^{-1}(\mathscr{E}_{X\to Y} \underset{\rho^{-1}\mathscr{E}_Y}{\otimes} \rho^{-1}\mathscr{N}).$$

We shall denote this by $f^*\mathscr{N}$ and call it the pull-back of $\mathscr{N}$. Then Theorem 8.8.1 reads as follows.

THEOREM 8.9.1. *Let U_X and U_Y be open subsets of T^*X and T^*Y, respectively. Let $\mathscr{N}$ be a coherent $(\mathscr{E}_Y|_U)$-module. Assume*

(i) $\rho_f^{-1}(\operatorname{Supp}\mathscr{N}) \cap \tilde{\omega}_f^{-1}(U_X) \to U_X$ *is a finite morphism.*

Then we have

(a) $\mathscr{T}or_j^{\rho_f^{-1}\mathscr{E}_Y}(\mathscr{E}_{X\to Y}, \mathscr{N}) = 0$ *for* $j \neq 0$.

(b) $\mathscr{M} = \tilde{\omega}_{f*}(\mathscr{E}_{X\to Y} \underset{\rho_f^{-1}\mathscr{E}_Y}{\otimes} \rho_f^{-1}\mathscr{N})|_{U_X}$ *is a coherent* $\mathscr{E}_X$*-module.*

(c) $\operatorname{Supp} M = \tilde{\omega}_f\rho_f^{-1}\operatorname{Supp}\mathscr{N} \cap U_X$.

8.10. Similarly let $g\colon Y \to X$ be a holomorphic map and let Δ_g be the graph of g, i.e. $\{(g(y), y) \in X \times Y; y \in Y\}$. Then we have the isomorphisms

$$\begin{array}{ccccc} T^*X & \overset{\rho_g}{\leftarrow} & Y \times T^*X & \overset{\tilde{\omega}_g}{\rightarrow} & T^*Y \\ \| & & \wr\| & & \|\ \mathrm{id}. \\ T^*X & \underset{p_1}{\leftarrow} & T^*_{\Delta_g}(X\times Y) & \underset{p_2}{\overset{a}{\rightarrow}} & T^*Y \end{array} \qquad (8.10.1)$$

We set $\mathscr{E}_{X\leftarrow Y} = \omega_Y \underset{\mathscr{O}_Y}{\otimes} \mathscr{C}_{\Delta_g|X\times Y}$ and regard this as a sheaf on $Y \underset{X}{\times} T^*X$. Then $\mathscr{E}_{X\leftarrow Y}$ is a $(\rho^{-1}\mathscr{E}_X, \tilde{\omega}^{-1}\mathscr{E}_Y)$-bi-module. For an $\mathscr{E}_Y$-module $\mathscr{N}$ we have

$$\int \mathscr{C}_{\Delta_g|X\times Y} \circ \mathscr{N} = \mathbf{R}\rho_*\tilde{\omega}^{-1}(\mathscr{E}_{X\leftarrow Y} \underset{\tilde{\omega}^{-1}\mathscr{E}_Y}{\otimes} \tilde{\omega}^{-1}\mathscr{N}).$$

We shall denote this by $\int_g \mathscr{N}$. Then Theorem 8.8.1 applies to this case and we have

THEOREM 8.10.1. *Let* U_X *and* U_Y *be open subsets of* T^*X *and* T^*Y, *respectively. Let* $\mathscr{N}$ *be a coherent* $(\mathscr{E}_Y|_{U_Y})$*-module. Assume*

(i) $\rho_g : \tilde{\omega}_g^{-1}(\mathrm{Supp}\,\mathscr{N}) \cap \rho_g^{-1}(U_X) \to U_X$ *is a finite morphism.*

Then we have

(a) $\mathscr{T}or_j^{\tilde{\omega}_g^{-1}\mathscr{E}_Y}(\mathscr{E}_{X\leftarrow Y}, \tilde{\omega}_g^{-1}\mathscr{N}) = 0 \quad \text{for} \quad j \neq 0.$

(b) $\mathscr{M} = \rho_{g*}(\mathscr{E}_{X\leftarrow Y} \underset{\tilde{\omega}_g^{-1}\mathscr{E}_Y}{\otimes} \tilde{\omega}_g^{-1}\mathscr{N})|_{U_X}$ *is a coherent* $\mathscr{E}_X|_{U_X}$*-module.*

(c) $\mathrm{Supp}\,\mathscr{M} = \rho_g(\tilde{\omega}_g^{-1}\,\mathrm{Supp}\,\mathscr{N} \cap U_X)$.

§ 9. REGULARITY CONDITIONS (See [KK], [K-O])

9.1. Let us recall the notion of regular singularity of ordinary differential equations. Let $P(x, \partial) = \sum_{j\leq m} a_j(x)\partial^j$ be a linear differential operator in one variable x. We assume that the $a_j(x)$ are holomorphic on a neighborhood of $x = 0$. Then we say that the origin 0 is a regular singularity of $Pu = 0$ if

$$(*) \qquad \mathrm{ord}_{x=0}a_j(x) \geqslant \mathrm{ord}_{x=0}a_m(x) - (m-j)\,.$$

Here $\mathrm{ord}_{x=0}$ means the order of the zero. In this case, the local structure of the equation is very simple. In fact, the $\mathscr{D}_X$-module $\mathscr{D}_X/\mathscr{D}_XP$ is a direct sum of copies of the following modules:

$$\mathscr{O}_X = \mathscr{D}_X/\mathscr{D}_X\partial,\ \mathscr{B}_{\{0\}|X} = \mathscr{D}_X/\mathscr{D}_Xx,\ \mathscr{D}_X/\mathscr{D}_X(x\partial-\lambda)^{m+1} \quad (\lambda\in\mathbf{C}, m\in\mathbf{N}),$$
$$\mathscr{D}_X/\mathscr{D}_X(x\partial)^{m+1}x \quad (m\in\mathbf{N}),\ \mathscr{D}_X/\mathscr{D}_X\partial(x\partial)^{m+1} \quad (m\in\mathbf{N}).$$

If we denote by u the canonical generator, then we have $Pu = 0$. By multiplying either a power of ∂ or a power of x, we obtain

$$\sum_{j=0}^{N} b_j(x)\,(x\partial)^j u = 0$$

with $b_N(x) = 1$. Hence $\mathscr{F} = \sum_{j=0}^{\infty} \mathscr{O}(x\partial)^j u = \sum_{j=0}^{N-1} \mathscr{O}(x\partial)^j u$ is a coherent $\mathscr{O}$-submodule of $\mathscr{M}$ which satisfies $(x\partial)\mathscr{F} \subset \mathscr{F}$. We shall generalize this property to the case of several variables.

9.2. Let X be a complex manifold, Ω an open subset of $\mathring{T}^*X$ and V a closed involutive complex submanifold of Ω. Let us define