

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 32 (1986)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: INTRODUCTION TO MICROLOCAL ANALYSIS
Autor: Kashiwara, Masaki
Kapitel: §8. FUNCTORIAL PROPERTIES OF MICRO-DIFFERENTIAL MODULES (See [SKK])
DOI: <https://doi.org/10.5169/seals-55089>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 14.12.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

$$\partial_x \mapsto \alpha \partial_x - \beta x \partial_t$$

$$x \mapsto \gamma \partial_x \partial_t^{-1} + \delta x$$

$$\partial_t \mapsto \partial_t$$

$$t \mapsto t + \frac{1}{2} \{ \langle \partial_x, {}^t \gamma \alpha \partial_x \rangle \partial_t^{-2} + \langle \partial_x, {}^t \gamma \beta x \rangle \partial_t^{-1} \\ + \langle {}^t \gamma \beta x, \partial_x \rangle \partial_t^{-1} + \langle x, {}^t \delta \beta x \rangle \}.$$

Then we have $\Psi_{g_1} \Psi_{g_2} = \Psi_{g_1 g_2}$.

§ 8. FUNCTORIAL PROPERTIES OF MICRO-DIFFERENTIAL MODULES (See [SKK])

8.1. External Tensor Product.

Let X and Y be complex manifolds and let p_1 and p_2 be the projections $T^*(X \times Y) \rightarrow T^*X$ and $T^*(X \times Y) \rightarrow T^*Y$, respectively. Then $\mathcal{E}_{X \times Y}$ contains $p_1^{-1} \mathcal{E}_X \otimes_{\mathbb{C}} p_2^{-1} \mathcal{E}_Y$ as a subring. For an \mathcal{E}_X -module \mathcal{M} and an \mathcal{E}_Y -module \mathcal{N} , we define the $\mathcal{E}_{X \times Y}$ -module $\mathcal{M} \hat{\otimes} \mathcal{N}$ by

$$(8.1.1) \quad \mathcal{M} \hat{\otimes} \mathcal{N} = \mathcal{E}_{X \times Y} \otimes_{p_1^{-1} \mathcal{E}_X \otimes_{\mathbb{C}} p_2^{-1} \mathcal{E}_Y} (p_1^{-1} \mathcal{M} \otimes_{\mathbb{C}} p_2^{-1} \mathcal{N}).$$

Then one can easily see

PROPOSITION 8.1.1.

- (i) $\mathcal{M} \hat{\otimes} \mathcal{N}$ is an exact functor in \mathcal{M} and in \mathcal{N} and $\text{Supp}(\mathcal{M} \hat{\otimes} \mathcal{N}) = \text{Supp} \mathcal{M} \times \text{Supp} \mathcal{N}$.
- (ii) If \mathcal{M} is \mathcal{E}_X -coherent and \mathcal{N} is \mathcal{E}_Y -coherent, then $\mathcal{M} \hat{\otimes} \mathcal{N}$ is $\mathcal{E}_{X \times Y}$ -coherent.

8.2. For a complex submanifold Y of a complex manifold X of codimension l , the sheaf $\lim_{\rightarrow m} \mathcal{E}xt_{\mathcal{O}_X}^l(\mathcal{O}_X/\mathcal{I}^m, \mathcal{O}_X)$ has a natural structure of \mathcal{D}_X -module,

which is denoted by $\mathcal{B}_{Y|X}$. Here \mathcal{I} is the defining ideal of Y . The homomorphism $\mathcal{O}_Y \rightarrow \mathcal{E}xt_{\mathcal{O}_X}^l(\mathcal{O}_Y, \Omega_X^l) \rightarrow \Omega_X^l \otimes_{\mathcal{O}_X} \mathcal{B}_{Y|X}$ gives the canonical section $c(Y, X)$ of $\Omega_X^l \otimes_{\mathcal{O}_X} \mathcal{B}_{Y|X}$. If we take local coordinates (x_1, \dots, x_n) of X such that Y is defined by $x_1 = \dots = x_l = 0$, then we have

$$\mathcal{B}_{Y|X} \cong \mathcal{D}_X / \sum_{j \leq l} \mathcal{D}_X x_j + \sum_{j > l} \mathcal{D}_X \partial_j.$$

If we denote by δ the canonical generator of the left hand side, then $c(Y, X)$ corresponds to $dx_1 \wedge \dots \wedge dx_l \otimes \delta$. We set

$$\mathcal{C}_{Y|X} = \mathcal{C}_X \otimes_{\pi^{-1}\mathcal{O}_X} \pi^{-1}\mathcal{B}_{Y|X}.$$

Therefore locally we have

$$\mathcal{C}_{Y|X} \cong \mathcal{C}_X / \sum_{j \leq d} \mathcal{C}_X x_j + \sum_{j > d} \mathcal{C}_X \partial_j.$$

Then $\mathcal{C}_{Y|X}$ is a coherent \mathcal{C}_X -module whose support is T_Y^*X .

8.3. For an invertible \mathcal{O}_X -module \mathcal{L} , $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{C}_X \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes -1}$ has a natural structure of sheaves of rings, by the composition rule

$$(s \otimes P \otimes s^{\otimes -1}) \circ (s \otimes Q \otimes s^{\otimes -1}) = s \otimes PQ \otimes s^{\otimes -1}$$

for an invertible section s of \mathcal{L} and $P, Q \in \mathcal{C}_X$.

Then the category $\text{Mod}(\mathcal{C}_X)$ of left \mathcal{C}_X -modules and the category $\text{Mod}(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{C}_X \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes -1})$ of left $(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{C}_X \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes -1})$ -modules are equivalent by the functor

$$\text{Mod}(\mathcal{C}_X) \ni \mathcal{M} \mapsto \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{M} \in \text{Mod}(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{C}_X \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes -1}).$$

8.4. Let ω_X be the canonical sheaf on X , i.e. the sheaf of differential forms with top degree. Let a be the antipodal map of T^*X , i.e. the multiplication by -1 . Then we have the anti-ring isomorphism.

$$(8.4.1) \quad \omega_X \otimes_{\mathcal{O}_X} \mathcal{C}_X \otimes_{\mathcal{O}_X} \omega_X^{\otimes -1} \xrightarrow{\sim} a^{-1}\mathcal{C}_X.$$

This homomorphism is given by using a local coordinate system (x_1, \dots, x_n) as follows. For $P = \sum P_j(x, \partial) \in \mathcal{C}_X$ we define $P^* = \sum P_j^*(x, \partial)$, called the formal adjoint of P ([SKK] Chap. II, Th. 1.5.1), by

$$(8.4.2) \quad P_i^*(x, -\xi) = \sum_{\substack{j=i-|\alpha| \\ \alpha \in \mathbb{N}^n}} \frac{(-1)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha \partial_x^\alpha P_j(x, \xi).$$

This is well-defined and satisfies

$$(8.4.3) \quad (P^*)^* = P$$

$$(8.4.4) \quad (PQ)^* = Q^*P^* .$$

Then the isomorphism (8.4.1) is given by

$$(8.4.5) \quad dx \otimes P \otimes (dx)^{\otimes -1} \mapsto P^*$$

where $dx = dx_1 \wedge \dots \wedge dx_n \in \omega_X$. This is independent of coordinate transformations.

8.5. The isomorphism (8.4.1) can be explained as follows. Let Δ_X be the diagonal set of $X \times X$, and let p_j be the j -th projection from $T_{\Delta_X}^*(X \times X)$ to T^*X for $j = 1, 2$. Then the p_j are isomorphisms and $p_2 \circ p_1^{-1} = a$. Let q_j be the j -th projection from $T^*(X \times X)$ to X ($j = 1, 2$). Then $c(\Delta_X, X \times X)$ gives the canonical section of $q_2^{-1}\omega_X \otimes_{q_2^{-1}\mathcal{O}_X} \mathcal{C}_{\Delta_X|X \times X}$. Since $\mathcal{C}_{\Delta_X|X \times X}$ is a $p_1^{-1}\mathcal{E}_X$ -module, this section gives a homomorphism

$$p_1^{-1}\mathcal{E}_X \rightarrow q_2^{-1}\omega_X \otimes_{q_2^{-1}\mathcal{O}_X} \mathcal{C}_{\Delta_X|X \times X} .$$

It turns out that this is an isomorphism and the right multiplication of \mathcal{O}_X on \mathcal{E}_X corresponds to the \mathcal{O}_X -module structure of $q_2^{-1}\omega_X \otimes_{q_2^{-1}\mathcal{O}_X} \mathcal{C}_{\Delta_X|X \times X}$ via q_2 . Thus we obtain

$$p_1^{-1}(\omega_X \otimes_{\mathcal{O}_X} \mathcal{E}_X \otimes_{\mathcal{O}_X} \omega_X^{\otimes -1}) \xrightarrow{\sim} q_1^{-1}\omega_X \otimes_{q_1^{-1}\mathcal{O}_X} \mathcal{C}_{\Delta_X|X \times X} .$$

This last being isomorphic to $p_2^{-1}\mathcal{E}_X$, we obtain

$$\omega_X \otimes_{\mathcal{O}_X} \mathcal{E}_X \otimes_{\mathcal{O}_X} \omega_X^{\otimes -1} \xrightarrow{\sim} p_1 p_2^{-1} \mathcal{E}_X \simeq a^{-1} \mathcal{E}_X .$$

8.6. By 8.3 and 8.4, if \mathcal{M} is a left $\mathcal{E}_{X|U}$ -module for an open set U of T^*X , then $\omega_X \otimes_{\mathcal{O}_X} a^{-1}\mathcal{M}$ is a right $(\mathcal{E}_{X|aU})$ -module.

8.7. For a left coherent \mathcal{E}_X -module \mathcal{M} , $\mathcal{E}xt_{\mathcal{E}_X}^j(\mathcal{M}, \mathcal{E}_X)$ is a right coherent \mathcal{E}_X -module. Therefore $\mathcal{E}xt_{\mathcal{E}_X}^j(\mathcal{M}, \mathcal{E}_X) \otimes_{\mathcal{O}_X} \omega_X^{\otimes -1}$ is a left \mathcal{E}_X -module by § 8.6.

If \mathcal{M} is holonomic then $\mathcal{E}xt_{\mathcal{E}_X}^j(\mathcal{M}, \mathcal{E}_X) = 0$ for $j \neq n = \dim X$ (See [SKK], [K1]). Set $\mathcal{M}^* = \mathcal{E}xt_{\mathcal{E}_X}^n(\mathcal{M}, \mathcal{E}_X) \otimes_{\mathcal{O}_X} \omega_X^{\otimes -1}$. Then \mathcal{M}^* is also a holonomic \mathcal{E}_X -module.

We call \mathcal{M}^* the dual system of \mathcal{M} . We have $\mathcal{M}^{**} = \mathcal{M}$, and $\mathcal{M} \mapsto \mathcal{M}^*$ is an exact contravariant functor on the category of holonomic \mathcal{E}_X -modules.

8.8. Let X and Y be complex manifolds, and let $p_1: T^*(X \times Y) \rightarrow T^*X$ and $p_2: T^*(X \times Y) \rightarrow T^*Y$ be the canonical projections. Let p_2^a denote $p_2 \circ a$. Let \mathcal{K} be a left $\mathcal{E}_{X \times Y}$ -module defined on an open subset Ω of $T^*(X \times Y)$. Then, by § 8.6, $\omega_Y \otimes_{\mathcal{O}_Y} \mathcal{K}$ has a structure of $(p_1^{-1}\mathcal{E}_X, p_2^{a-1}\mathcal{E}_Y)$ -bi-module. For an \mathcal{E}_Y -module \mathcal{N} ,

$$\mathcal{M} = p_{1*}((\omega_Y \otimes_{\mathcal{O}_Y} \mathcal{K}) \otimes_{p_2^{a-1}\mathcal{E}_Y} p_2^{a-1}\mathcal{N})$$

has a structure of \mathcal{E}_X -module. We have the following

THEOREM 8.8.1. *Let Ω, U_X and U_Y be open subsets of $T^*(X \times Y), T^*X$ and T^*Y , respectively. Let \mathcal{K} be a coherent $(\mathcal{E}_{X \times Y}|_{\Omega})$ -module and \mathcal{N} a coherent $(\mathcal{E}_Y|_{U_Y})$ -module. Assume*

(i) $p_1: p_1^{-1}U_X \cap \text{Supp } \mathcal{K} \cap p_2^{a-1} \text{Supp } \mathcal{N} \rightarrow U_X$ is a finite morphism.

Then we have

(a) $\mathcal{F}or_j^{p_2^{a-1}\mathcal{E}_Y} (\omega_Y \otimes_{\mathcal{O}_Y} \mathcal{K}, p_2^{a-1}\mathcal{N})|_{p_1^{-1}U_X} = 0$ for $j \neq 0$.

(b) $\mathcal{M} = p_{1*}((\omega_Y \otimes_{\mathcal{O}_Y} \mathcal{K}) \otimes_{p_2^{a-1}\mathcal{E}_Y} p_2^{a-1}\mathcal{N})|_{U_X}$ is a coherent \mathcal{E}_X -module.

(c) $\text{Supp } \mathcal{M} = U_X \cap p_1(\text{Supp } \mathcal{K} \cap p_2^{a-1} \text{Supp } \mathcal{N})$.

We denote $p_{1*}((\omega_Y \otimes_{\mathcal{O}_Y} \mathcal{K}) \otimes_{p_2^{a-1}\mathcal{E}_Y} p_2^{a-1}\mathcal{N})$ by $\int_Y \mathcal{K} \circ \mathcal{N}$.

8.9. Let $f: X \rightarrow Y$ be a holomorphic map and let Δ_f be the graph of f , i.e. $\{(x, f(x)) \in X \times Y; x \in X\}$, then $\mathcal{K} = \mathcal{C}_{\Delta_f|_{X \times Y}}$ is a coherent $\mathcal{E}_{X \times Y}$ -module whose support is $T_{\Delta_f}^*(X \times Y)$. Now let $\tilde{\omega}$ be the canonical map $X \times T^*_Y Y \rightarrow T^*X$ and ρ the projection $X \times T^*_Y Y \rightarrow T^*Y$. Then we have the following diagram

$$(8.9.1) \quad \begin{array}{ccccc} T^*X & \xleftarrow{\tilde{\omega}_f} & X \times T^*_Y Y & \xrightarrow{\rho_f} & T^*Y \\ \text{id} \parallel & & \wr & & \parallel \text{id} \\ T^*X & \xleftarrow{p_1} & T_{\Delta_f}^*(X \times Y) & \xrightarrow{p_2^a} & T^*Y \end{array}$$

We set $\mathcal{E}_{X \rightarrow Y} = \omega_Y \otimes_{\mathcal{O}_Y} \mathcal{C}_{\Delta_f|X \times Y}$ and consider this as a sheaf on $X \times T^*Y$ by the above isomorphism. Then $\mathcal{E}_{X \rightarrow Y}$ is a $(\tilde{\omega}^{-1}\mathcal{E}_X, \rho^{-1}\mathcal{E}_Y)$ -bi-module. For an \mathcal{E}_Y -module \mathcal{N} ,

$$\int \mathcal{K} \circ \mathcal{N} = \mathbf{R}\tilde{\omega}_* \rho^{-1}(\mathcal{E}_{X \rightarrow Y} \otimes_{\rho^{-1}\mathcal{E}_Y} \rho^{-1}\mathcal{N}).$$

We shall denote this by $f^*\mathcal{N}$ and call it the pull-back of \mathcal{N} . Then Theorem 8.8.1 reads as follows.

THEOREM 8.9.1. *Let U_X and U_Y be open subsets of T^*X and T^*Y , respectively. Let \mathcal{N} be a coherent $(\mathcal{E}_Y|_U)$ -module. Assume*

(i) $\rho_f^{-1}(\text{Supp } \mathcal{N}) \cap \tilde{\omega}_f^{-1}(U_X) \rightarrow U_X$ is a finite morphism.

Then we have

(a) $\mathcal{T}or_j^{\rho_f^{-1}\mathcal{E}_Y}(\mathcal{E}_{X \rightarrow Y}, \mathcal{N}) = 0$ for $j \neq 0$.

(b) $\mathcal{M} = \tilde{\omega}_{f*}(\mathcal{E}_{X \rightarrow Y} \otimes_{\rho_f^{-1}\mathcal{E}_Y} \rho_f^{-1}\mathcal{N})|_{U_X}$ is a coherent \mathcal{E}_X -module.

(c) $\text{Supp } \mathcal{M} = \tilde{\omega}_f \rho_f^{-1} \text{Supp } \mathcal{N} \cap U_X$.

8.10. Similarly let $g: Y \rightarrow X$ be a holomorphic map and let Δ_g be the graph of g , i.e. $\{(g(y), y) \in X \times Y; y \in Y\}$. Then we have the isomorphisms

$$(8.10.1) \quad \begin{array}{ccccc} T^*X & \xleftarrow{p_g} & Y \times T^*X & \xrightarrow{\tilde{\omega}_g} & T^*Y \\ \parallel & & \wr & & \parallel \text{ id.} \\ T^*X & \xleftarrow{p_1} & T^*_{\Delta_g}(X \times Y) & \xrightarrow[p_2]{\tilde{\omega}_g} & T^*Y. \end{array}$$

We set $\mathcal{E}_{X \leftarrow Y} = \omega_Y \otimes_{\mathcal{O}_Y} \mathcal{C}_{\Delta_g|X \times Y}$ and regard this as a sheaf on $Y \times T^*X$. Then $\mathcal{E}_{X \leftarrow Y}$ is a $(\rho^{-1}\mathcal{E}_X, \tilde{\omega}^{-1}\mathcal{E}_Y)$ -bi-module. For an \mathcal{E}_Y -module \mathcal{N} we have

$$\int \mathcal{C}_{\Delta_g|X \times Y} \circ \mathcal{N} = \mathbf{R}\rho_* \tilde{\omega}^{-1}(\mathcal{E}_{X \leftarrow Y} \otimes_{\tilde{\omega}^{-1}\mathcal{E}_Y} \tilde{\omega}^{-1}\mathcal{N}).$$

We shall denote this by $\int_g \mathcal{N}$. Then Theorem 8.8.1 applies to this case and we have

THEOREM 8.10.1. *Let U_X and U_Y be open subsets of T^*X and T^*Y , respectively. Let \mathcal{N} be a coherent $(\mathcal{E}_Y|_{U_Y})$ -module. Assume*

(i) $\rho_g: \tilde{\omega}_g^{-1}(\text{Supp } \mathcal{N}) \cap \rho_g^{-1}(U_X) \rightarrow U_X$ *is a finite morphism.*

Then we have

(a) $\mathcal{T}or_j^{\tilde{\omega}_g^{-1}\mathcal{E}_Y}(\mathcal{E}_{X \leftarrow Y}, \tilde{\omega}_g^{-1}\mathcal{N}) = 0$ *for* $j \neq 0$.

(b) $\mathcal{M} = \rho_{g*}(\mathcal{E}_{X \leftarrow Y} \otimes_{\tilde{\omega}_g^{-1}\mathcal{E}_Y} \tilde{\omega}_g^{-1}\mathcal{N})|_{U_X}$ *is a coherent* $\mathcal{E}_X|_{U_X}$ -*module.*

(c) $\text{Supp } \mathcal{M} = \rho_g(\tilde{\omega}_g^{-1} \text{Supp } \mathcal{N} \cap U_X)$.

§ 9. REGULARITY CONDITIONS (See [KK], [K-O])

9.1. Let us recall the notion of regular singularity of ordinary differential equations. Let $P(x, \partial) = \sum_{j \leq m} a_j(x)\partial^j$ be a linear differential operator in one variable x . We assume that the $a_j(x)$ are holomorphic on a neighborhood of $x = 0$. Then we say that the origin 0 is a regular singularity of $Pu = 0$ if

$$(*) \quad \text{ord}_{x=0} a_j(x) \geq \text{ord}_{x=0} a_m(x) - (m - j).$$

Here $\text{ord}_{x=0}$ means the order of the zero. In this case, the local structure of the equation is very simple. In fact, the \mathcal{D}_X -module $\mathcal{D}_X/\mathcal{D}_X P$ is a direct sum of copies of the following modules:

$$\begin{aligned} \mathcal{O}_X &= \mathcal{D}_X/\mathcal{D}_X \partial, \mathcal{B}_{\{0\}|X} = \mathcal{D}_X/\mathcal{D}_X x, \mathcal{D}_X/\mathcal{D}_X (x\partial - \lambda)^{m+1} \quad (\lambda \in \mathbb{C}, m \in \mathbb{N}), \\ &\mathcal{D}_X/\mathcal{D}_X (x\partial)^{m+1} x \quad (m \in \mathbb{N}), \mathcal{D}_X/\mathcal{D}_X \partial (x\partial)^{m+1} \quad (m \in \mathbb{N}). \end{aligned}$$

If we denote by u the canonical generator, then we have $Pu = 0$. By multiplying either a power of ∂ or a power of x , we obtain

$$\sum_{j=0}^N b_j(x) (x\partial)^j u = 0$$

with $b_N(x) = 1$. Hence $\mathcal{F} = \sum_{j=0}^{\infty} \mathcal{O}(x\partial)^j u = \sum_{j=0}^{N-1} \mathcal{O}(x\partial)^j u$ is a coherent \mathcal{O} -submodule of \mathcal{M} which satisfies $(x\partial)\mathcal{F} \subset \mathcal{F}$. We shall generalize this property to the case of several variables.

9.2. Let X be a complex manifold, Ω an open subset of T^*X and V a closed involutive complex submanifold of Ω . Let us define