

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 32 (1986)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** INTRODUCTION TO MICROLOCAL ANALYSIS  
**Autor:** Kashiwara, Masaki  
**Kapitel:** §6. MICRO-DIFFERENTIAL OPERATORS AND THE SYMPLECTIC STRUCTURE ON THE COTANGENT BUNDLE  
**DOI:** <https://doi.org/10.5169/seals-55089>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 19.04.2026

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

## § 5. THE VANISHING CYCLE SHEAF

5.1. Let  $M$  be a real manifold and  $f: M \rightarrow \mathbf{R}$  a continuous map. For a sheaf  $\mathcal{F}$  on  $M$ ,  $\mathcal{H}_{f^{-1}(\mathbf{R}^+)}^j(\mathcal{F})|_{f^{-1}(0)}$  is called the ( $j$ -th) *vanishing cycle sheaf* of  $\mathcal{F}$ . Here  $\mathbf{R}^+ = \{t \in \mathbf{R}; t \geq 0\}$ . This measures how the cohomology groups of  $\mathcal{F}$  change across the fibers of  $f$ . Its algebro-geometric version is studied by Grothendieck-Deligne ([D]).

5.2. Let  $(X, \mathcal{O}_X)$  be a complex manifold. Let  $f: X \rightarrow \mathbf{R}$  be a  $C^\infty$ -map and consider the vanishing cycle sheaf  $\mathcal{H}_{f^{-1}(\mathbf{R}^+)}^j(\mathcal{O}_X)|_{f^{-1}(0)}$ . Let  $s$  be the section of  $f^{-1}(0) \rightarrow T^*X$  given by  $df$ . Then we have

PROPOSITION 5.2.1 ([KS1] § 3, [K2] § 4.2).  $\mathcal{H}_{f^{-1}(\mathbf{R}^+)}^j(\mathcal{O}_X)|_{f^{-1}(0)}$  has a structure of an  $s^{-1}\mathcal{E}_X$ -module.

Let  $P$  be a differential operator. If  $\sigma(P)$  does not vanish on  $s(f^{-1}(0))$ , then  $P$  has an inverse in  $s^{-1}\mathcal{E}_X$  by Proposition 2.2.3. Therefore we obtain

COROLLARY 5.2.2. If  $\sigma(P)|_{s(f^{-1}(0))} \neq 0$ , then

$$P: \mathcal{H}_{f^{-1}(\mathbf{R}^+)}^j(\mathcal{O}_X)|_{f^{-1}(0)} \rightarrow \mathcal{H}_{f^{-1}(\mathbf{R}^+)}^j(\mathcal{O}_X)|_{f^{-1}(0)}$$

is bijective.

5.3. More generally, let  $\mathcal{M}$  be a coherent  $\mathcal{D}_X$ -module, and set

$$\mathcal{F}^\bullet = \mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X).$$

Then the preceding corollary shows that

$$\mathbf{R}\Gamma_{f^{-1}(\mathbf{R}^+)}(\mathcal{F}^\bullet)|_{f^{-1}(0)} = 0 \quad \text{if} \quad s(f^{-1}(0)) \cap \text{Ch}(\mathcal{M}) = \emptyset.$$

Here  $\text{Ch}\mathcal{M}$  denotes the characteristic variety of  $\mathcal{M}$ .

5.4. To consider vanishing cycle sheaves is very near to the "microlocal" consideration. In this direction, see [K-S2].

## § 6. MICRO-DIFFERENTIAL OPERATORS AND THE SYMPLECTIC STRUCTURE ON THE COTANGENT BUNDLE

6.1. The ring  $\mathcal{E}_X$  is a non-commutative ring. This fact gives rise to new phenomena which are not shared by commutative rings such as the ring of

holomorphic functions. They are also closely related to the symplectic structure of the cotangent bundle.

6.2. Let us recall the symplectic structure on the cotangent bundle.

Let  $\theta_X$  denote the canonical 1-form on the cotangent bundle  $T^*X$  of a complex manifold. Then  $d\theta_X$  gives the symplectic structure on  $T^*X$ . The Hamiltonian map  $H: T^*(T^*X) \xrightarrow{\sim} T(T^*X)$  is given by

$$(6.1.1) \quad \begin{aligned} \langle \eta, v \rangle &= \langle d\theta_X, v \wedge H(\eta) \rangle \quad \text{for } \eta \in T^*(T^*X) \\ \text{and } v &\in T(T^*X). \end{aligned}$$

For a function  $f$  on  $T^*X$ ,  $H(df)$  is denoted by  $H_f$  and the Poisson bracket  $\{f, g\}$  is defined as  $H_f(g)$ . If we denote by  $\mathcal{X}$  the Euler vector field (i.e. the infinitesimal action of  $\mathbf{C}^*$  on  $T^*X$ ), then we have

$$\mathcal{X} = H(-\theta_X).$$

With a local coordinate system  $(x_1, \dots, x_n)$  of  $X$  and the associated local coordinate system  $(x_1, \dots, x_n; \xi_1, \dots, \xi_n)$  of  $T^*X$ , we have

$$\theta_X = \sum \xi_j dx_j,$$

$$d\theta_X = \sum d\xi_j dx_j,$$

$$H: d\xi_j \mapsto \partial/\partial x_j, dx_j \mapsto -\partial/\partial \xi_j$$

$$\mathcal{X} = \sum \xi_j \frac{\partial}{\partial \xi_j},$$

$$\{f, g\} = \sum \left( \frac{\partial f}{\partial \xi_j} \frac{\partial g}{\partial x_j} - \frac{\partial g}{\partial \xi_j} \frac{\partial f}{\partial x_j} \right).$$

6.3. This structure is deeply related to the ring of micro-differential operators. The first relation between them appears in the following

PROPOSITION 6.3.1. For  $P \in \mathcal{E}(\lambda)$  and  $Q \in \mathcal{E}(\mu)$ , set

$$[P, Q] = PQ - QP \in \mathcal{E}(\lambda + \mu - 1).$$

Then

$$\sigma_{\lambda + \mu - 1}([P, Q]) = \{\sigma_\lambda(P), \sigma_\mu(Q)\}.$$

An analytic subset  $V$  of  $T^*X$  is called *involutive* if  $f|_V = g|_V = 0$  implies  $\{f, g\}|_V = 0$ .

The following theorem exhibits a phenomenon which has no analogue in the commutative case.

**THEOREM 6.3.2 ([G]).** *Let  $\mathcal{M}$  be a coherent  $\mathcal{E}_X$ -module defined on an open subset  $\Omega$  of  $T^*X$  and let  $\mathcal{L}$  be a  $\mathcal{E}_X(0)|_\Omega$ -module which is a union of coherent  $\mathcal{E}_X(0)$ -modules. Then  $V = \{p \in \Omega; \mathcal{L} \text{ is not coherent over } \mathcal{E}_X(0) \text{ on any neighborhood of } p\}$  is an involutive analytic subset of  $\Omega$ .*

**COROLLARY 6.3.3 ([SKK] Chap. II, Theorem 5.3.2, [M]).** *For any coherent  $\mathcal{E}_X$ -module  $\mathcal{M}$ ,  $\text{Supp } \mathcal{M}$  is involutive.*

Since any involutive subset has codimension less than or equal to  $\dim X$ , we have

**COROLLARY 6.3.4.** *The support of a coherent  $\mathcal{E}_X$ -module has codimension  $\leq \dim X$ .*

After some algebraic calculation, this implies

**THEOREM 6.3.5 ([SKK] Chap. II, Theorem 5.3.5).** *For any point  $p \in T^*X$ ,  $\mathcal{E}_{X,p}$  has a global cohomological dimension  $\dim X$ .*

6.4. An analytic subset  $\Lambda$  of  $T^*X$  is called *Lagrangian* if  $\Lambda$  is involutive and  $\dim \Lambda = \dim X$ . A coherent  $\mathcal{E}_X$ -module is called *holonomic* if its support is Lagrangian.

## § 7. QUANTIZED CONTACT TRANSFORMATIONS

7.1. In the previous section, we saw that the symplectic structure of  $T^*X$  is closely related to micro-differential operators via the relation of commutator and Poisson bracket. In this section, we shall explain another relation.

*Definition 7.2.1.* Let  $X$  and  $Y$  be complex manifolds of the same dimension. A morphism  $\varphi$  from an open subset  $U$  of  $T^*X$  to  $T^*Y$  is called a *homogeneous symplectic transformation* if  $\varphi^*\theta_Y = \theta_X$ .