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then

$$\lambda\left(\frac{N}{5} \pm 1\right) = \frac{\lambda(N \pm 5)}{\lambda(5)} = 1 = -\lambda(N) = \lambda\left(\frac{N}{5}\right),$$

and  $N/5 = 4n/5 \in I_n$  is good. We may therefore suppose that at least one of values  $\lambda(N+5)$  and  $\lambda(N-5)$  equals 1.

For definiteness we shall assume  $\lambda(N+5) = 1$ ; the other case is treated in exactly the same way.

If  $\lambda(N+3) = 1$  or  $\lambda(N+6) = 1$ , then  $N+4 \in I_n$  or  $N+5 \in I_n$  is good. But in the remaining case

$$\lambda(N+3) = \lambda(N+6) = -1$$

we have

$$\lambda\left(\frac{N}{3}\right) = \lambda\left(\frac{N}{3} + 1\right) = \lambda\left(\frac{N}{3} + 2\right) = 1,$$

so that  $(N+3)/3 \in I_n$  is good.

Thus (3) implies the existence of a good integer in the interval (4), as we had to show.

#### 4. PROOF OF THE THEOREM, CONCLUSION

So far we have proved that (1) has infinitely many solutions in the cases  $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 1$  and  $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = -1$ . But this obviously implies that for each of the triples  $(\varepsilon_1, \varepsilon_2, \varepsilon_3) = (1, 1, -1), (-1, -1, 1), (1, -1, -1)$  and  $(-1, 1, 1)$  there are also infinitely many solutions to (1). It remains therefore to consider the triples  $(1, -1, 1)$  and  $(-1, 1, -1)$ . Since the arguments in both cases are the same (with  $+1$  and  $-1$  interchanged), we shall confine ourselves to the case  $(\varepsilon_1, \varepsilon_2, \varepsilon_3) = (1, -1, 1)$ . Accordingly, we call  $n \geq 2$  good, whenever

$$\lambda(n+1) = \lambda(n-1) = 1, \quad \lambda(n) = -1.$$

We have to show that there are infinitely many such  $n$ .

Suppose, to get a contradiction, that there are only finitely many good integers, all of them  $\leq N_0$ , say. Suppose further that

$$(5) \quad \lambda(n) = 1 (m_0 \leq n \leq n_0)$$

holds for some integers  $n_0 > m_0 \geq 2N_0$ . We shall show that then

$$(6) \quad \lambda(n) = 1(m_i \leq n \leq n_i)$$

holds for all  $i \geq 0$ , where  $m_i$  and  $n_i$  are defined inductively by

$$(7) \quad m_{i+1} = \left\lceil \frac{3m_i + 1}{2} \right\rceil, \quad n_{i+1} = \left\lceil \frac{3n_i}{2} \right\rceil (i \geq 0).$$

This will easily lead to the desired contradiction.

By our assumption (5), (6) holds for  $i = 0$ . Assume now that (6) does not hold for all  $i \geq 0$ , and let  $i \geq 0$  be minimal such that (6) holds for  $i$  and fails for  $i + 1$ . Thus, for some  $n \in [m_{i+1}, n_{i+1}]$ , which we shall fix, we have  $\lambda(n) = -1$ . Write

$$(8) \quad 2n = 3n' + \theta (\theta \in \{0, 1, -1\}).$$

From (7) we get

$$3m_i \leq 2m_{i+1} \leq 2n \leq 2n_{i+1} \leq 3n_i,$$

so that

$$m_i \leq n' \leq n_i,$$

and hence by (6) (which we assumed to hold for  $i$ )

$$\lambda(3n') = -\lambda(n') = -1.$$

Since, by our assumption  $\lambda(n) = -1$ ,

$$\lambda(2n) = -\lambda(n) = 1,$$

we cannot have  $\theta = 0$  in (8). The arguments in the cases  $\theta = \pm 1$  being identical, we shall henceforth assume that (8) holds with  $\theta = 1$ .

We must have

$$\lambda(2(n-1)) = \lambda(3n'-1) = -1,$$

since otherwise  $3n'$  would be good and

$$3n' \geq 3m_i \geq 3m_0 > N_0,$$

in contradiction to our assumption. Also, since

$$m_i \leq n' + 1 = \frac{2}{3}(n+1) \leq \left\lceil \frac{2}{3}(n_{i+1}+1) \right\rceil = \left\lceil \frac{2}{3} \left( \left\lceil \frac{3n_i}{2} \right\rceil + 1 \right) \right\rceil \leq n_i,$$

we have by (6)

$$\lambda(2(n+1)) = \lambda(3(n'+1)) = -\lambda(n'+1) = -1.$$

These two identities imply

$$\lambda(n \pm 1) = -\lambda(2(n \pm 1)) = 1,$$

and since  $\lambda(n) = -1$ , we conclude that  $n(>N_0)$  is good and therefore arrive at a contradiction.

Thus (5) (with  $n_0 > m_0 \geq 2N_0$ ) implies (6) for all  $i \geq 0$ . To derive from this the desired contradiction, we suppose first that (5) holds for some  $n_0 > m_0 \geq 2N_0$  satisfying

$$(9) \quad n_0 - m_0 \geq 3.$$

In other words, we suppose (for the moment) that there exist four consecutive integers  $n \geq 2N_0$ , for which  $\lambda(n) = 1$ . Putting  $d_i = n_i - m_i$ , we have, by the recursion formulae (7),

$$d_{i+1} \geq \frac{3}{2} d_i - 1 = \frac{3}{2} d_i \left(1 - \frac{2}{3d_i}\right) \quad (i \geq 0).$$

Taking into account (9), we obtain by induction in turn

$$d_i \geq 3 \quad (i \geq 0),$$

$$d_i \geq 3 \left(\frac{7}{6}\right)^i \quad (i \geq 0),$$

and finally

$$d_i \geq \left(\frac{3}{2}\right)^i \prod_{j=0}^i \left(1 - \frac{2}{3d_j}\right) \geq C \left(\frac{3}{2}\right)^i \quad (i \geq 0),$$

where

$$C = \prod_{j \geq 0} \left(1 - \frac{2}{9} \left(\frac{6}{7}\right)^j\right) > 0.$$

Since on the other hand by (7)

$$d_i \leq n_i \leq \left(\frac{3}{2}\right)^i n_0 \quad (i \geq 0),$$

we see from (6), that there are arbitrary large values of  $x$ , such that  $\lambda(n)$  is constant in the interval  $[x(1-\varepsilon), x]$ , where  $\varepsilon = C/n_0$ . But this is impossible since, for  $x$  sufficiently large, every such interval contains integers  $n$  and  $n'$  of the form

$$n = 4^a 9^b, \quad n' = 2 \cdot 4^c 9^d (a, b, c, d \in \mathbf{N}),$$

for which

$$\lambda(n) = 1, \quad \lambda(n') = -1.$$

We therefore have obtained the desired contradiction under the assumption that there exist four consecutive integers  $n \geq 2N_0$ , for which  $\lambda(n) = 1$ . By the part of the theorem already proved, there exist at least three such integers. Therefore (5) holds for some  $m_0 > 2N_0$  with  $n_0 = m_0 + 2$ , and we may now assume that

$$\lambda(m_0 - 1) = \lambda(m_0 + 3) = -1.$$

If  $m_0$  is odd, then this implies

$$\lambda\left(\frac{m_0 - 1}{2}\right) = \lambda\left(\frac{m_0 + 3}{2}\right) = 1, \quad \lambda\left(\frac{m_0 + 1}{2}\right) = -1,$$

so that  $(m_0 + 1)/2 > N_0$  is good, in contradiction to our assumption. But if  $m_0$  is even, then defining  $m_1$  and  $n_1$  by (7), (6) holds for  $i = 1$ , and we have

$$m_1 \geq 2N_0, \quad n_1 - m_1 = \frac{3(m_0 + 2)}{2} - \frac{3m_0}{2} = 3.$$

Thus we are back in the case already treated.

By contradiction, we therefore conclude that (1) has infinitely many solutions for  $(\varepsilon_1, \varepsilon_2, \varepsilon_3) = (1, -1, 1)$ , and the proof of the theorem is complete.

## 5. CONCLUDING REMARKS

In the foregoing proof, the relevant property of the Liouville function was that  $\lambda(n)$  is completely multiplicative and assumes only the values  $\pm 1$ . Besides this, we used only the fact that  $\lambda(2) = \lambda(3) = \lambda(5) = -1$  and (in the proof of the lemma)

$$\lambda(14) = \lambda(16) = 1, \quad \lambda(29) = \lambda(31) = -1.$$

The proof, as it stands, works for any completely multiplicative function  $f(n) = \pm 1$  with these properties. By suitably modifying the proof, it is possible to cover other classes of multiplicative functions as well.