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Kapitel: 2. A Lemma

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desired result. The drawback of this method is that it gives no indication on how to settle the general case of the conjecture, or even the case $k = 4$. It seems that for this completely new ideas are needed, and Chowla's remark on the difficulty of the problem appears to be justified, as far as the general form of the conjecture is concerned.

2. A Lemma

Lemma. *Each of the equations*

$$\lambda(15n-1) = \lambda(15n+1) = 1$$

and

$$\lambda(15n-1) = \lambda(15n+1) = -1$$

holds for infinitely many positive integers n.

Proof. Given a positive integer $n_0 \geqslant 2$, define $n_i, i \geqslant 1$, inductively by

$$n_{i+1} = n_i(4n_i^2-3) \quad (i \geqslant 0).$$

It is easily checked that

$$n_{i+1} \pm 1 = (n_i \pm 1)(2n_i \mp 1)^2 \quad (i \geqslant 0),$$

so that

$$\lambda(n_{i+1} \pm 1) = \lambda(n_i \pm 1) = \ldots = \lambda(n_0 \pm 1) \quad (i \geqslant 0).$$

Also, it follows by induction that $n_0 \mid n_i$ for all $i \geqslant 0$. Therefore, taking in turn $n_0 = 15$ and $n_0 = 30$ and noting that

$$\lambda(14) = \lambda(16) = 1, \quad \lambda(29) = \lambda(31) = -1,$$

we obtain two infinite sequences $(n_i^{(+)})$ and $(n_i^{(-)})$ with the required properties

$$n_i^{(\pm)} \equiv 0 \pmod{15}, \quad \lambda(n_i^{(+)} \pm 1) = 1, \quad \lambda(n_i^{(-)} \pm 1) = -1.$$

Remark. The same argument shows that for any completely multiplicative function f assuming only the values ± 1 and for given $\varepsilon_1, \varepsilon_2 = \pm 1$ and $a \geqslant 2$, the system

$$n \equiv 0 \pmod{a}, \quad f(n-1) = \varepsilon_1, \quad f(n+1) = \varepsilon_2$$

has infinitely many solutions, provided it has at least one solution. It would be interesting to have an analogous result for three (or more) consecutive values, but the above method does not work in this case.

3. Proof of the Theorem, beginning

We shall show here that each of the equations

$$\lambda(n) = \lambda(n+1) = \lambda(n-1) = 1 \tag{2}$$

and

$$\lambda(n) = \lambda(n+1) = \lambda(n-1) = -1 \tag{2$'$}$$

has infinitely many solutions. Since the arguments for the two cases are completely symmetric, we shall carry out the proof only in the case of equation (2).

Call an integer $n \geqslant 2$ "good", if (2) holds for this n. We have to show that there are infinitely many good integers. To this end we shall show that for any positive integer n satisfying

$$n \equiv 0 \pmod{15}, \quad \lambda(n+1) = \lambda(n-1) = 1, \tag{3}$$

the interval

$$I_n = \left[\frac{4n}{5}, 4n + 5\right] \tag{4}$$

contains a good integer. Since by the lemma (3) holds for infinitely many positive integers n, the desired result follows.

To prove our assertion we fix a positive integer n, for which (3) holds. We may suppose $\lambda(n) = -1$, since otherwise $n \in I_n$ is good, and we are done. Put $N = 4n$, and note that, by construction, N is divisible by 3, 4 and 5. From (3) we get, using the multiplicativity of the function λ,

$$\lambda(N \pm 4) = \lambda\big(4(n \pm 1)\big) = \lambda(4)\,\lambda(n \pm 1) = 1,$$

and our assumption $\lambda(n) = -1$ implies

$$\lambda(N) = \lambda(4n) = \lambda(4)\,\lambda(n) = -1.$$

If now

$$\lambda(N+5) = \lambda(N-5) = -1,$$