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ON CONSECUTIVE VALUES OF THE LIOUVILLE FUNCTION

by Adolf HILDEBRAND

ABSTRACT: It is shown that for every choice of $\varepsilon_i = \pm 1, i = 1, 2, 3$, there exist infinitely many positive integers n, such that $\lambda(n+i) = \varepsilon_i$, $i = 1, 2, 3$, where λ denotes the Liouville function. [1])

1. INTRODUCTION

Let $\lambda(n)$ denote the Liouville function, i.e. $\lambda(n) = +1$ or -1 according as the number of prime factors of n (counted with multiplicity) is even or odd. It is natural to expect that the sequence $(\lambda(n))$ behaves like a random sequence of $\pm$ signs. A particularly attractive and highly plausible conjecture is that every finite "block" of $\pm$ signs occurs in this sequence infinitely often, i.e. for any given numbers $\varepsilon_i = \pm 1, 1 \leqslant i \leqslant k$, there are infinitely many integers $n \geqslant 1$, such that

$$\lambda(n+i) = \varepsilon_i \quad (1 \leqslant i \leqslant k).$$

Whereas for $k = 1$ and $k = 2$ this conjecture holds trivially, there are no results known in the literature for larger values of k. In [1, p. 95, problem 56], Chowla states the above conjecture and remarks that "for $k \geqslant 3$, this seems an extremely hard conjecture". The purpose of this paper is to prove the conjecture in the first non-trivial case $k = 3$.

THEOREM. *For any choice of* $\varepsilon_i = \pm 1, i = 1, 2, 3$, *there are infinitely many positive integers* n *such that*

$$\lambda(n+i) = \varepsilon_i \quad (i=1, 2, 3). \tag{1}$$

We shall use for the proof an "ad hoc" method, which leads in a relatively simple way and using only very elementary arguments to the

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desired result. The drawback of this method is that it gives no indication on how to settle the general case of the conjecture, or even the case $k = 4$. It seems that for this completely new ideas are needed, and Chowla's remark on the difficulty of the problem appears to be justified, as far as the general form of the conjecture is concerned.

2. A Lemma

Lemma. *Each of the equations*

$$\lambda(15n-1) = \lambda(15n+1) = 1$$

and

$$\lambda(15n-1) = \lambda(15n+1) = -1$$

holds for infinitely many positive integers n.

Proof. Given a positive integer $n_0 \geqslant 2$, define n_i, $i \geqslant 1$, inductively by

$$n_{i+1} = n_i(4n_i^2-3) \quad (i \geqslant 0) .$$

It is easily checked that

$$n_{i+1} \pm 1 = (n_i \pm 1)(2n_i \mp 1)^2 \quad (i \geqslant 0) ,$$

so that

$$\lambda(n_{i+1} \pm 1) = \lambda(n_i \pm 1) = \ldots = \lambda(n_0 \pm 1) \quad (i \geqslant 0) .$$

Also, it follows by induction that $n_0 \mid n_i$ for all $i \geqslant 0$. Therefore, taking in turn $n_0 = 15$ and $n_0 = 30$ and noting that

$$\lambda(14) = \lambda(16) = 1 , \quad \lambda(29) = \lambda(31) = -1 ,$$

we obtain two infinite sequences $(n_i^{(+)})$ and $(n_i^{(-)})$ with the required properties

$$n_i^{(\pm)} \equiv 0 \pmod{15} , \quad \lambda(n_i^{(+)} \pm 1) = 1 , \quad \lambda(n_i^{(-)} \pm 1) = -1 .$$

Remark. The same argument shows that for any completely multiplicative function f assuming only the values ± 1 and for given $\varepsilon_1, \varepsilon_2 = \pm 1$ and $a \geqslant 2$, the system

$$n \equiv 0 \pmod{a} , \quad f(n-1) = \varepsilon_1 , \quad f(n+1) = \varepsilon_2$$