

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 31 (1985)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** THE TRACE AS AN ALGEBRA HOMOMORPHISM  
**Autor:** Osborn, Howard  
**Kapitel:** 2. The trace  
**DOI:** <https://doi.org/10.5169/seals-54566>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 02.09.2025

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

1.1 *Definition*: The *third product* of any two elements  $\mathbf{A}$  and  $\mathbf{B}$  of  $\Pi_p \text{End } \wedge^p V$  is given by  $\mathbf{A} \times \mathbf{B} = \alpha^{-1}((\alpha\mathbf{A})(\alpha\mathbf{B})) \in \Pi_p \text{End } \wedge^p V$ , where  $(\alpha\mathbf{A})(\alpha\mathbf{B})$  is the composition product of the shuffle products  $\alpha\mathbf{A} = e \cdot I \cdot \mathbf{A}$  and  $\alpha\mathbf{B} = e \cdot I \cdot \mathbf{B}$ .

Since the composition product is associative the third product is trivially associative. Furthermore, if  $I_0 \in \text{End } \wedge^0 V$  represents the unit element in  $\Pi_p \text{End } \wedge^p V$  with respect to the shuffle product one has

$$I_0 \times \mathbf{A} = \alpha^{-1}((\alpha I_0)(\alpha\mathbf{A})) = \alpha^{-1}((e \cdot I)(\alpha\mathbf{A})) = \alpha^{-1}(\mathbf{I}(\alpha\mathbf{A})) = \alpha^{-1}(\alpha\mathbf{A}) = \mathbf{A}$$

and similarly  $\mathbf{A} \times I_0 = \mathbf{A}$  for any  $\mathbf{A} \in \Pi_p \text{End } \wedge^p V$ ; that is,  $I_0$  is also the unit element of  $\Pi_p \text{End } \wedge^p V$  with respect to the third product. The rationale for introducing the third product appears in the next section.

## 2. THE TRACE

We now specialize the arbitrary  $R$ -module  $V$  of the preceding section.

2.1 *Definition*: A module  $V$  over a commutative ring  $R$  with unit is *traceable* of rank  $n > 0$  if and only if  $\text{End } \wedge^n V$  is a free  $R$ -module of rank one.

If  $\wedge^n V$  is itself free of rank one then  $V$  is clearly traceable of rank  $n$ . However,  $\text{End } \wedge^n V$  can be free of rank one with no such condition on  $\wedge^n V$ . For example, let  $X$  be any paracompact hausdorff space, let  $R$  be the ring  $C(X)$  of continuous real-valued functions on  $X$ , and let  $V$  be the  $C(X)$ -module of continuous sections of a real  $n$ -plane bundle  $\xi$  over  $X$ ; then  $V$  is traceable of rank  $n$ . However  $\wedge^n V$  is itself free of rank one if and only if  $\xi$  is orientable.

Flanders [1] showed for any module  $V$  over a commutative ring with unit that if  $\wedge^n V$  is free of rank one then  $\wedge^p V = 0$  for every  $p > n$ ; a similar argument shows that if  $V$  is traceable of rank  $n > 0$  then  $\text{End } \wedge^p V = 0$  for every  $p > n$ . Thus if  $V$  is traceable of rank  $n > 0$  there is no distinction between the direct product  $\Pi_p \text{End } \wedge^p V$  and the direct sum  $\amalg_p \text{End } \wedge^p V$ . Consequently the third product of Definition 1.1 can be regarded as a product in  $\amalg_p \text{End } \wedge^p V$  whenever  $V$  is traceable.

If  $V$  is traceable of rank  $n$  then every element of  $\text{End } \wedge^n V$  is scalar multiplication by a unique element of the commutative ground ring  $R$  with unit. For example, for any  $\mathbf{A} \in \amalg_p \text{End } \wedge^p V$  and each  $p = 0, \dots, n$  let

$(\alpha\mathbf{A})_p \in \text{End } \wedge^p V$  be the  $p^{\text{th}}$  component of  $\alpha\mathbf{A} \in \prod_p \text{End } \wedge^p V$ . Then  $(\alpha\mathbf{A})_n \in \text{End } \wedge^n V$  is scalar multiplication by a unique element of  $R$ .

2.2 *Definition*: If  $V$  is a traceable module of rank  $n > 0$  over a commutative ground ring  $R$  with unit, the *trace* of any  $\mathbf{A} \in \prod_p \text{End } \wedge^p V$  is the unique element  $\text{tr } \mathbf{A} \in R$  such that  $(\alpha\mathbf{A})_n = (\text{tr } \mathbf{A})I_n \in \text{End } \wedge^n V$ , for the identity endomorphism  $I_n \in \text{End } \wedge^n V$ .

For example, if  $A \in \text{End } V$  then  $(\alpha A)_n = A \cdot I_{n-1}$  for the identity endomorphism  $I_{n-1} \in \text{End } \wedge^{n-1} V$ . One easily verifies that if  $V$  is a free  $R$ -module of rank  $n$  then the classical trace of  $A$  is precisely that element  $\text{tr } A \in R$  such that  $A \cdot I_{n-1} = (\text{tr } A)I_n \in \text{End } \wedge^n V$ .

2.3 THEOREM. Let  $\prod_p \text{End } \wedge^p V$  be the endomorphism algebra generated by the endomorphisms of a traceable module  $V$ , multiplication being the third product; then the trace is an algebra homomorphism  $\prod_p \text{End } \wedge^p V \xrightarrow{\text{tr}} R$  over the ground ring  $R$ . Specifically, both  $\text{tr}(\mathbf{A} + \mathbf{B}) = \text{tr } \mathbf{A} + \text{tr } \mathbf{B}$  and  $\text{tr}(\mathbf{A} \times \mathbf{B}) = (\text{tr } \mathbf{A})(\text{tr } \mathbf{B})$  for any elements  $\mathbf{A}$  and  $\mathbf{B}$  of  $\prod_p \text{End } \wedge^p V$ .

*Proof*. Additivity of the trace is trivial. To show that the trace also respects the third product suppose that  $V$  is traceable of rank  $n$ , and let  $(\alpha\mathbf{A})_p, (\alpha\mathbf{B})_p$  and  $\alpha(\mathbf{A} \times \mathbf{B})_p$  denote the components of  $\alpha\mathbf{A}$ ,  $\alpha\mathbf{B}$  and  $\alpha(\mathbf{A} \times \mathbf{B})$  in  $\text{End } \wedge^p V$  for each  $p = 0, \dots, n$ . By the definition  $\mathbf{A} \times \mathbf{B} = \alpha^{-1}((\alpha\mathbf{A})(\alpha\mathbf{B}))$  of the third product one has  $\alpha(\mathbf{A} \times \mathbf{B}) = (\alpha\mathbf{A})(\alpha\mathbf{B})$  for the composition product  $(\alpha\mathbf{A})(\alpha\mathbf{B})$ , that is,  $\prod_p \alpha(\mathbf{A} \times \mathbf{B})_p = \prod_p (\alpha\mathbf{A})_p (\alpha\mathbf{B})_p$ . In particular  $\alpha(\mathbf{A} \times \mathbf{B})_n = (\alpha\mathbf{A})_n (\alpha\mathbf{B})_n$  in the  $n^{\text{th}}$  component  $\text{End } \wedge^n V$ , so that

$$\text{tr}(\mathbf{A} \times \mathbf{B})I_n = ((\text{tr } \mathbf{A})I_n)((\text{tr } \mathbf{B})I_n) = (\text{tr } \mathbf{A})(\text{tr } \mathbf{B})I_n$$

by definition of the trace; since  $\text{End } \wedge^n V$  is free on the single generator  $I_n$  this implies  $\text{tr}(\mathbf{A} \times \mathbf{B}) = (\text{tr } \mathbf{A})(\text{tr } \mathbf{B})$  as claimed.

### 3. PROPERTIES OF THE THIRD PRODUCT

We now establish several properties of the third product. Although these properties do not require the  $R$ -module  $V$  to be traceable, we shall later impose a condition on elements of the  $R$ -module  $\prod_r \text{End } \wedge^r V$  itself; the condition will automatically be satisfied in the applications.

Let  $V$  be any module over a commutative ring  $R$  with unit, and let  $\mathbf{A}$  and  $\mathbf{B}$  be elements of the direct product  $\prod_r \text{End } \wedge^r V$  whose only