1.1 *Definition:* The *third product* of any two elements $\mathbf{A}$ and $\mathbf{B}$ of Π_p End $\wedge^p V$ is given by $\mathbf{A} \times \mathbf{B} = \alpha^{-1}((\alpha\mathbf{A})(\alpha\mathbf{B})) \in \Pi_p$ End $\wedge^p V$, where $(\alpha\mathbf{A})(\alpha\mathbf{B})$ is the composition product of the shuffle products $\alpha\mathbf{A} = e^{\cdot I} \cdot \mathbf{A}$ and $\alpha\mathbf{B} = e^{\cdot I} \cdot \mathbf{B}$.

Since the composition product is associative the third product is trivially associative. Furthermore, if $I_0 \in$ End $\wedge^0 V$ represents the unit element in Π_p End $\wedge^p V$ with respect to the shuffle product one has

$$I_0 \times \mathbf{A} = \alpha^{-1}((\alpha I_0)(\alpha\mathbf{A})) = \alpha^{-1}((e^{\cdot I})(\alpha\mathbf{A})) = \alpha^{-1}(\mathbf{I}(\alpha\mathbf{A})) = \alpha^{-1}(\alpha\mathbf{A}) = \mathbf{A}$$

and similarly $\mathbf{A} \times I_0 = \mathbf{A}$ for any $\mathbf{A} \in \Pi_p$ End $\wedge^p V$; that is, I_0 is also the unit element of Π_p End $\wedge^p V$ with respect to the third product. The rationale for introducing the third product appears in the next section.

2. The trace

We now specialize the arbitrary R-module V of the preceding section.

2.1 *Definition:* A module V over a commutative ring R with unit is *traceable* of rank $n > 0$ if and only if End $\wedge^n V$ is a free R-module of rank one.

If $\wedge^n V$ is itself free of rank one then V is clearly traceable of rank n. However, End $\wedge^n V$ can be free of rank one with no such condition on $\wedge^n V$. For example, let X be any paracompact hausdorff space, let R be the ring $C(X)$ of continuous real-valued functions on X, and let V be the $C(X)$-module of continuous sections of a real n-plane bundle ξ over X; then V is traceable of rank n. However $\wedge^n V$ is itself free of rank one if and only if ξ is orientable.

Flanders [1] showed for any module V over a commutative ring with unit that if $\wedge^n V$ is free of rank one then $\wedge^p V = 0$ for every $p > n$; a similar argument shows that if V is traceable of rank $n > 0$ then End $\wedge^p V = 0$ for every $p > n$. Thus if V is traceable of rank $n > 0$ there is no distinction between the direct product Π_p End $\wedge^p V$ and the direct sum $\amalg_p$ End $\wedge^p V$. Consequently the third product of Definition 1.1 can be regarded as a product in $\amalg_p$ End $\wedge^p V$ whenever V is traceable.

If V is traceable of rank n then every element of End $\wedge^n V$ is scalar multiplication by a unique element of the commutative ground ring R with unit. For example, for any $\mathbf{A} \in \amalg_p$ End $\wedge^p V$ and each $p = 0, \ldots, n$ let

$(\alpha\mathbf{A})_p \in \text{End} \wedge^p V$ be the p^{th} component of $\alpha\mathbf{A} \in \amalg_p \text{End} \wedge^p V$. Then $(\alpha\mathbf{A})_n \in \text{End} \wedge^n V$ is scalar multiplication by a unique element of R.

2.2 *Definition:* If V is a traceable module of rank $n > 0$ over a commutative ground ring R with unit, the *trace* of any $\mathbf{A} \in \amalg_p \text{End} \wedge^p V$ is the unique element $\text{tr}\,\mathbf{A} \in R$ such that $(\alpha\mathbf{A})_n = (\text{tr}\,\mathbf{A})I_n \in \text{End} \wedge^n V$, for the identity endomorphism $I_n \in \text{End} \wedge^n V$.

For example, if $A \in \text{End}\, V$ then $(\alpha A)_n = A \cdot I_{n-1}$ for the identity endomorphism $I_{n-1} \in \text{End} \wedge^{n-1} V$. One easily verifies that if V is a free R-module of rank n then the classical trace of A is precisely that element $\text{tr}\, A \in R$ such that $A \cdot I_{n-1} = (\text{tr}\, A)I_n \in \text{End} \wedge^n V$.

2.3 THEOREM. *Let* $\amalg_p \text{End} \wedge^p V$ *be the endomorphism algebra generated by the endomorphisms of a traceable module* V, *multiplication being the third product; then the trace is an algebra homomorphism* $\amalg_p \text{End} \wedge^p V \xrightarrow{\text{tr}} R$ *over the ground ring* R. *Specifically, both* $\text{tr}(\mathbf{A}+\mathbf{B}) = \text{tr}\,\mathbf{A} + \text{tr}\,\mathbf{B}$ *and* $\text{tr}(\mathbf{A}\times\mathbf{B}) = (\text{tr}\,\mathbf{A})(\text{tr}\,\mathbf{B})$ *for any elements* $\mathbf{A}$ *and* $\mathbf{B}$ *of* $\amalg_p \text{End} \wedge^p V$.

Proof. Additivity of the trace is trivial. To show that the trace also respects the third product suppose that V is traceable of rank n, and let $(\alpha\mathbf{A})_p, (\alpha\mathbf{B})_p$ and $\alpha(\mathbf{A}\times\mathbf{B})_p$ denote the components of $\alpha\mathbf{A}$, $\alpha\mathbf{B}$ and $\alpha(\mathbf{A}\times\mathbf{B})$ in $\text{End} \wedge^p V$ for each $p = 0, \ldots, n$. By the definition $\mathbf{A} \times \mathbf{B} = \alpha^{-1}((\alpha\mathbf{A})(\alpha\mathbf{B}))$ of the third product one has $\alpha(\mathbf{A}\times\mathbf{B}) = (\alpha\mathbf{A})(\alpha\mathbf{B})$ for the composition product $(\alpha\mathbf{A})(\alpha\mathbf{B})$, that is, $\amalg_p \alpha(\mathbf{A}\times\mathbf{B})_p = \amalg_p(\alpha\mathbf{A})_p(\alpha\mathbf{B})_p$. In particular $\alpha(\mathbf{A}\times\mathbf{B})_n = (\alpha\mathbf{A})_n(\alpha\mathbf{B})_n$ in the n^{th} component $\text{End} \wedge^n V$, so that

$$\text{tr}(\mathbf{A}\times\mathbf{B})I_n = \big((\text{tr}\,\mathbf{A})I_n\big)\big((\text{tr}\,\mathbf{B})I_n\big) = (\text{tr}\,\mathbf{A})(\text{tr}\,\mathbf{B})I_n$$

by definition of the trace; since $\text{End} \wedge^n V$ is free on the single generator I_n this implies $\text{tr}(\mathbf{A}\times\mathbf{B}) = (\text{tr}\,\mathbf{A})(\text{tr}\,\mathbf{B})$ as claimed.

3. PROPERTIES OF THE THIRD PRODUCT

We now establish several properties of the third product. Although these properties do not require the R-module V to be traceable, we shall later impose a condition on elements of the R-module $\Pi_r \text{End} \wedge^r V$ itself; the condition will automatically be satisfied in the applications.

Let V be any module over a commutative ring R with unit, and let $\mathbf{A}$ and $\mathbf{B}$ be elements of the direct product $\Pi_r \text{End} \wedge^r V$ whose only