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THE TRACE AS AN ALGEBRA HOMOMORPHISM

by Howard OSBORN

0. INTRODUCTION

Let A be any endomorphism of an appropriately restricted module V over a commutative ring with unit. The coefficients of the characteristic polynomial of A are the *elementary invariants* of A, being *traces* of A-induced endomorphisms of the exterior powers $\wedge^p V$. Similarly the *sums-of-powers invariants* of A are *traces* of the compositions $A, AA, \ldots$ of A with itself. For example, if V is free of rank n and A is represented by a diagonal matrix with diagonal entries $t_1, \ldots, t_n$, then the elementary invariants and the sums-of-powers invariants are the usual elementary symmetric polynomials $\sigma_1, \ldots, \sigma_n$ and sums-of-powers polynomials $s_1, s_2, \ldots$, respectively, in $t_1, \ldots, t_n$. Since $s_1, s_2, \ldots$ can be expressed as the Newton polynomials in $\sigma_1, \ldots, \sigma_n$ one can easily use an appropriate "splitting principle" to prove that the sums-of-powers invariants of any endomorphism A of an appropriately restricted module V are the Newton polynomials in the elementary invariants of the same endomorphism A. The technique applies equally well to other trace-induced invariants of A.

In this intentionally elementary note such relations among the invariants of A are presented from a different point of view as images under the trace of identities in a new endomorphism algebra associated to the module V. Specifically, if End $\wedge^p V$ denotes the module of endomorphisms of the p^{th} exterior power $\wedge^p V$ of V, then one can provide the direct sum $\amalg_p \operatorname{End} \wedge^p V$ with a new product for which the trace becomes an *algebra* homomorphism onto the ground ring, preserving products as well as sums. There are universal identities in $\amalg_p \operatorname{End} \wedge^p V$ which express relations among the various endomorphisms induced by any endomorphism A of V itself, and one applies the trace to obtain corresponding identities among the invariants of A in the ground ring. The Newton identities are presented in this form to illustrate the technique.

MR subject classification (1980): 16A65 Endomorphism rings.

1. Endomorphism algebras

In this section V will be an arbitrary module over a commutative ring R with unit, and for each $p \geqslant 0$ $\wedge^p V$ will be its p^{th} exterior power and $\text{End} \wedge^p V$ will be the R-module of endomorphisms $\wedge^p V \to \wedge^p V$; $\Pi_p \text{End} \wedge^p V$ will be the direct product of the R-modules $\text{End} \wedge^p V$. We shall define three distinct products in $\Pi_p \text{End} \wedge^p V$; the first two products are standard, and they will be used to define the third product. If $\wedge^p V$ itself vanishes for sufficiently large $p \geqslant 0$ the direct product $\Pi_p \text{End} \wedge^p V$ and the direct sum $\amalg_p \text{End} \wedge^p V$ agree; although this special condition will be satisfied in later sections the definitions in this section will be formulated in complete generality for the direct product $\Pi_p \text{End} \wedge^p V$.

Elements of $\Pi_p \text{End} \wedge^p V$ will be indicated by boldface capital letters $\mathbf{A}, \mathbf{B}, \ldots$, the p^{th} components being $A_p, B_p, \ldots \in \text{End} \wedge^p V$ for each $p \geqslant 0$. The simplest product in $\Pi_p \text{End} \wedge^p V$ is induced by compositions: the p^{th} component of the *composition product* $\mathbf{AB} \in \Pi_p \text{End} \wedge^p V$ is the usual composition $A_p B_p \in \text{End} \wedge^p V$ of the endomorphisms A_p and B_p of $\wedge^p V$, where $A_p B_q = 0$ for $p \neq q$. Trivially $\Pi_p \text{End} \wedge^p V$ is an associative R-algebra with respect to the composition product, and there is a two-sided unit element $\mathbf{I}$ whose p^{th} component is the identity endomorphism $I_p \in \text{End} \wedge^p V$ for each $p \geqslant 0$.

There is another reasonably familiar product in $\Pi_p \text{End} \wedge^p V$, the product of $A_p \in \text{End} \wedge^p V$ and $B_q \in \text{End} \wedge^q V$ providing an element

$$A_p \cdot B_q \in \text{End} \wedge^{p+q} V$$

for each $p \geqslant 0$ and each $q \geqslant 0$. Since elements of $\text{End} \wedge^{p+q} V$ are uniquely defined in terms of the behavior on exterior products $x_1 \wedge \ldots \wedge x_{p+q} \in \wedge^{p+q} V$, it suffices to require that

$$(A_p \cdot B_q)(x_1 \wedge \ldots \wedge x_{p+q}) = \sum_\pi \varepsilon_\pi A_p(x_{\pi 1} \wedge \ldots \wedge x_{\pi p}) \wedge B_q(x_{\pi(p+1)} \wedge \ldots \wedge x_{\pi(p+q)})$$

where the sum is computed over all permutations π of $\{1, \ldots, p+q\}$ such that both $\pi 1 < \ldots < \pi p$ and $\pi(p+1) < \ldots < \pi(p+q)$, and where ε_π is the parity ± 1 of the permutation π. Such "shuffle products" $A_p \cdot B_q \in \text{End} \wedge^{p+q} V$ provide a unique *shuffle product* $\mathbf{A} \cdot \mathbf{B} \in \Pi_r \text{End} \wedge^r V$ of any two elements $\mathbf{A}$ and $\mathbf{B}$ in $\Pi_r \text{End} \wedge^r V$.

One easily verifies that the shuffle product is associative and strictly commutative; specifically, $A_p \cdot B_q = B_q \cdot A_p \in \text{End} \wedge^{p+q} V$ with no plus-orminus signs. For example, for $p = 1$ and $q = 1$ one has

$$(A_1 \cdot B_1)(x_1 \wedge x_2) = A_1x_1 \wedge B_1x_2 - A_1x_2 \wedge B_1x_1$$
$$= - B_1x_2 \wedge A_1x_1 + B_1x_1 \wedge A_1x_2 = (B_1 \cdot A_1)(-x_2 \wedge x_1)$$
$$= (B_1 \cdot A_1)(x_1 \wedge x_2),$$

hence $A_1 \cdot B_1 = B_1 \cdot A_1 \in \text{End} \wedge^2 V$. The algebra $\Pi_p \text{End} \wedge^p V$ has a unique (two-sided) unit element with respect to the shuffle product, whose only nonzero component is the identity endomorphism I_0 of $\wedge^0 V$.

For any endomorphism A of V itself and any $p \geqslant 0$ there is a well-defined element $A_p \in \text{End} \wedge^p V$ such that

$$A_p(x_1 \wedge \dots \wedge x_p) = Ax_1 \wedge \dots \wedge Ax_p$$

for any $x_1 \wedge \dots \wedge x_p \in \wedge^p V$; in particular $A_1 = A$. Observe that the p-fold shuffle product $A^{\cdot p} = A \cdot \dots \cdot A$ is defined by

$$A^{\cdot p}(x_1 \wedge \dots \wedge x_p) = \textstyle\sum_\pi \varepsilon_\pi Ax_{\pi 1} \wedge \dots \wedge Ax_{\pi p},$$

the summation extending overall $p!$ permutations π of $\{1, \dots, p\}$. Since $\varepsilon_\pi Ax_{\pi 1} \wedge \dots \wedge Ax_{\pi p} = Ax_1 \wedge \dots \wedge Ax_p$ for each permutation π it follows that $A^{\cdot p} = p!\, A_p$. For this reason A_p can reasonably be written $\frac{1}{p!} A^{\cdot p}$, without requiring the ground ring to contain the element $\frac{1}{p!}$. Thus the direct product of the elements $A_p \left(= \frac{1}{p!} A^{\cdot p} \right)$ over all $p \geqslant 0$ is essentially an exponential $e^{\cdot A} \in \Pi_p \text{End} \wedge^p V$. One easily verifies that $e^{\cdot A} \cdot e^{\cdot(-A)} = I_0 = e^{\cdot(-A)} \cdot e^{\cdot A}$, where $I_0 \in \text{End} \wedge^0 V$ represents the unit element in $\Pi_p \text{End} \wedge^p V$ with respect to the shuffle product.

For each $p \geqslant 0$ the p-fold shuffle product $I^{\cdot p}$ of the identity endomorphism $I \in \text{End}\, V$ satisfies $\frac{1}{p!} I^{\cdot p} = I_p$, where I_p is the identity endomorphism in $\text{End} \wedge^p V$. Hence $e^{\cdot I}$ is precisely the two-sided unit element $\mathbf{I}$ of $\Pi_p \text{End} \wedge^p V$ with respect to the composition product. Since

$$e^{\cdot I} \cdot e^{\cdot(-I)} = I_0 = e^{\cdot(-I)} \cdot e^{\cdot I},$$

where $I_0 \in \text{End} \wedge^0 V$ represents the unit element with respect to the shuffle product, one can therefore define an invertible map α of $\Pi_p \text{End} \wedge^p V$ into itself by letting $\alpha\mathbf{A} \in \Pi_p \text{End} \wedge^p V$ be the shuffle product $e^{\cdot I} \cdot \mathbf{A}$ for any $\mathbf{A} \in \Pi_p \text{End} \wedge^p V$; the inverse α^{-1} of α is given by $\alpha^{-1}\mathbf{A} = e^{\cdot(-I)} \cdot \mathbf{A}$.

1.1 *Definition:* The *third product* of any two elements $\mathbf{A}$ and $\mathbf{B}$ of $\Pi_p \operatorname{End} \wedge^p V$ is given by $\mathbf{A} \times \mathbf{B} = \alpha^{-1}((\alpha\mathbf{A})(\alpha\mathbf{B})) \in \Pi_p \operatorname{End} \wedge^p V$, where $(\alpha\mathbf{A})(\alpha\mathbf{B})$ is the composition product of the shuffle products $\alpha\mathbf{A} = e^{\cdot I} \cdot \mathbf{A}$ and $\alpha\mathbf{B} = e^{\cdot I} \cdot \mathbf{B}$.

Since the composition product is associative the third product is trivially associative. Furthermore, if $I_0 \in \operatorname{End} \wedge^0 V$ represents the unit element in $\Pi_p \operatorname{End} \wedge^p V$ with respect to the shuffle product one has

$$I_0 \times \mathbf{A} = \alpha^{-1}((\alpha I_0)(\alpha\mathbf{A})) = \alpha^{-1}((e^{\cdot I})(\alpha\mathbf{A})) = \alpha^{-1}(\mathbf{I}(\alpha\mathbf{A})) = \alpha^{-1}(\alpha\mathbf{A}) = \mathbf{A}$$

and similarly $\mathbf{A} \times I_0 = \mathbf{A}$ for any $\mathbf{A} \in \Pi_p \operatorname{End} \wedge^p V$; that is, I_0 is also the unit element of $\Pi_p \operatorname{End} \wedge^p V$ with respect to the third product. The rationale for introducing the third product appears in the next section.

2. The trace

We now specialize the arbitrary R-module V of the preceding section.

2.1 *Definition:* A module V over a commutative ring R with unit is *traceable* of rank $n > 0$ if and only if $\operatorname{End} \wedge^n V$ is a free R-module of rank one.

If $\wedge^n V$ is itself free of rank one then V is clearly traceable of rank n. However, $\operatorname{End} \wedge^n V$ can be free of rank one with no such condition on $\wedge^n V$. For example, let X be any paracompact hausdorff space, let R be the ring $C(X)$ of continuous real-valued functions on X, and let V be the $C(X)$-module of continuous sections of a real n-plane bundle ξ over X; then V is traceable of rank n. However $\wedge^n V$ is itself free of rank one if and only if ξ is orientable.

Flanders [1] showed for any module V over a commutative ring with unit that if $\wedge^n V$ is free of rank one then $\wedge^p V = 0$ for every $p > n$; a similar argument shows that if V is traceable of rank $n > 0$ then $\operatorname{End} \wedge^p V = 0$ for every $p > n$. Thus if V is traceable of rank $n > 0$ there is no distinction between the direct product $\Pi_p \operatorname{End} \wedge^p V$ and the direct sum $\amalg_p \operatorname{End} \wedge^p V$. Consequently the third product of Definition 1.1 can be regarded as a product in $\amalg_p \operatorname{End} \wedge^p V$ whenever V is traceable.

If V is traceable of rank n then every element of $\operatorname{End} \wedge^n V$ is scalar multiplication by a unique element of the commutative ground ring R with unit. For example, for any $\mathbf{A} \in \amalg_p \operatorname{End} \wedge^p V$ and each $p = 0, \ldots, n$ let

$(\alpha\mathbf{A})_p \in \mathrm{End} \wedge^p V$ be the p^{th} component of $\alpha\mathbf{A} \in \amalg_p \mathrm{End} \wedge^p V$. Then $(\alpha\mathbf{A})_n \in \mathrm{End} \wedge^n V$ is scalar multiplication by a unique element of R.

2.2 *Definition:* If V is a traceable module of rank $n > 0$ over a commutative ground ring R with unit, the *trace* of any $\mathbf{A} \in \amalg_p \mathrm{End} \wedge^p V$ is the unique element $\mathrm{tr}\,\mathbf{A} \in R$ such that $(\alpha\mathbf{A})_n = (\mathrm{tr}\,\mathbf{A})I_n \in \mathrm{End} \wedge^n V$, for the identity endomorphism $I_n \in \mathrm{End} \wedge^n V$.

For example, if $A \in \mathrm{End}\, V$ then $(\alpha A)_n = A \cdot I_{n-1}$ for the identity endomorphism $I_{n-1} \in \mathrm{End} \wedge^{n-1} V$. One easily verifies that if V is a free R-module of rank n then the classical trace of A is precisely that element $\mathrm{tr}\, A \in R$ such that $A \cdot I_{n-1} = (\mathrm{tr}\, A)I_n \in \mathrm{End} \wedge^n V$.

2.3 THEOREM. *Let* $\amalg_p \mathrm{End} \wedge^p V$ *be the endomorphism algebra generated by the endomorphisms of a traceable module* V, *multiplication being the third product; then the trace is an algebra homomorphism* $\amalg_p \mathrm{End} \wedge^p V \xrightarrow{\mathrm{tr}} R$ *over the ground ring* R. *Specifically, both* $\mathrm{tr}(\mathbf{A}+\mathbf{B}) = \mathrm{tr}\,\mathbf{A} + \mathrm{tr}\,\mathbf{B}$ *and* $\mathrm{tr}(\mathbf{A}\times\mathbf{B}) = (\mathrm{tr}\,\mathbf{A})\,(\mathrm{tr}\,\mathbf{B})$ *for any elements* $\mathbf{A}$ *and* $\mathbf{B}$ *of* $\amalg_p \mathrm{End} \wedge^p V$.

Proof. Additivity of the trace is trivial. To show that the trace also respects the third product suppose that V is traceable of rank n, and let $(\alpha\mathbf{A})_p, (\alpha\mathbf{B})_p$ and $\alpha(\mathbf{A}\times\mathbf{B})_p$ denote the components of $\alpha\mathbf{A}$, $\alpha\mathbf{B}$ and $\alpha(\mathbf{A}\times\mathbf{B})$ in $\mathrm{End} \wedge^p V$ for each $p = 0, \ldots, n$. By the definition $\mathbf{A} \times \mathbf{B} = \alpha^{-1}((\alpha\mathbf{A})\,(\alpha\mathbf{B}))$ of the third product one has $\alpha(\mathbf{A}\times\mathbf{B}) = (\alpha\mathbf{A})\,(\alpha\mathbf{B})$ for the composition product $(\alpha\mathbf{A})\,(\alpha\mathbf{B})$, that is, $\amalg_p\, \alpha(\mathbf{A}\times\mathbf{B})_p = \amalg_p(\alpha\mathbf{A})_p\,(\alpha\mathbf{B})_p$. In particular $\alpha(\mathbf{A}\times\mathbf{B})_n = (\alpha\mathbf{A})_n\,(\alpha\mathbf{B})_n$ in the n^{th} component $\mathrm{End} \wedge^n V$, so that

$$\mathrm{tr}(\mathbf{A}\times\mathbf{B})I_n = \big((\mathrm{tr}\,\mathbf{A})I_n\big)\big((\mathrm{tr}\,\mathbf{B})I_n\big) = (\mathrm{tr}\,\mathbf{A})\,(\mathrm{tr}\,\mathbf{B})I_n$$

by definition of the trace; since $\mathrm{End} \wedge^n V$ is free on the single generator I_n this implies $\mathrm{tr}(\mathbf{A}\times\mathbf{B}) = (\mathrm{tr}\,\mathbf{A})\,(\mathrm{tr}\,\mathbf{B})$ as claimed.

3. PROPERTIES OF THE THIRD PRODUCT

We now establish several properties of the third product. Although these properties do not require the R-module V to be traceable, we shall later impose a condition on elements of the R-module $\Pi_r \mathrm{End} \wedge^r V$ itself; the condition will automatically be satisfied in the applications.

Let V be any module over a commutative ring R with unit, and let $\mathbf{A}$ and $\mathbf{B}$ be elements of the direct product $\Pi_r \mathrm{End} \wedge^r V$ whose only nonvanishing components occur in degrees p and q, respectively; for convenience we write A_p and B_q in place of $\mathbf{A}$ and $\mathbf{B}$. Recall that the identity endomorphism of each exterior power $\wedge^r V$ is given by $I_r = \frac{1}{r!} I^{\cdot r} \in \operatorname{End} \wedge^r V$ for the identity endomorphism $I = I_1 \in \operatorname{End} V$, and that there is a two-sided unit element $\mathbf{I} = e^{\cdot I} \in \Pi_r \operatorname{End} \wedge^r V$ with respect to composition products.

3.1 LEMMA. *For each* $r \geqslant 0$ *let* $(A_p \times B_q)_r \in \operatorname{End} \wedge^r V$ *be the* r^{th} *component of* $A_p \times B_q$; *then*

$$(A_p \times B_q)_r = \textstyle\sum_s (-1)^s (A_p \cdot I_{r-s-p})(B_q \cdot I_{r-s-q}) \cdot I_s ,$$

where $I_t = 0$ *for* $t < 0$.

Proof. This is an immediate consequence of the definition

$$\mathbf{A} \times \mathbf{B} = \alpha^{-1}((\alpha\mathbf{A})(\alpha\mathbf{B})), \quad \text{where} \quad \alpha\mathbf{A} = e^{\cdot I} \cdot \mathbf{A}, \alpha\mathbf{B} = e^{\cdot I} \cdot \mathbf{B},$$

and $\alpha^{-1}\mathbf{C} = e^{\cdot(-I)} \cdot \mathbf{C}$.

3.2 LEMMA. $(A_p \times B_q)_r = 0$ *for* $r < \max(p, q)$, *and if* $p = q = r$ *then* $(A_r \times B_r)_r$ *is the composition* $A_r B_r \in \operatorname{End} \wedge^r V$.

Proof. Immediate consequence of Lemma 3.1.

One can probably also use Lemma 3.1 directly to obtain the following more interesting properties of the third product: $(A_p \times B_q)_r = 0$ for $r > p + q$, and $(A_p \times B_q)_{p+q}$ is the shuffle product $A_p \cdot B_q \in \operatorname{End} \wedge^{p+q} V$. However, in order to avoid cumbersome computations we prove these results only for somewhat restricted endomorphisms $A_p \in \operatorname{End} \wedge^p V$ and $B_q \in \operatorname{End} \wedge^q V$.

3.3 *Definition:* An element $\mathbf{A} \in \Pi_r \operatorname{End} \wedge^r V$ is *one-generated* whenever each component is an R-linear combination of shuffle products of endomorphisms of V itself.

Clearly sums and all products of one-generated elements are one-generated; thus the one-generated elements form a subalgebra of $\Pi_r \operatorname{End} \wedge^r V$, with respect to any of the three products.

3.4 LEMMA. *For any one-generated elements* $\mathbf{B}$ *and* $\mathbf{C}$ *of* $\Pi_r \operatorname{End} \wedge^r V$ *and any* $A \in \operatorname{End} V$ *one has* $(\alpha A)(\mathbf{B} \cdot \mathbf{C}) = (\alpha A)\mathbf{B} \cdot \mathbf{C} + \mathbf{B} \cdot (\alpha A)\mathbf{C}$.

Proof. One may as well choose $\mathbf{B}$ and $\mathbf{C}$ to be shuffle products

$$B_1 \cdot \ldots \cdot B_p \in \operatorname{End} \wedge^p V \quad \text{and} \quad C_1 \cdot \ldots \cdot C_q \in \operatorname{End} \wedge^q V$$

of endomorphisms $B_1, \ldots, B_p, C_1, \ldots, C_q$ of V itself. Then

$$(\alpha A)\mathbf{B} = (A \cdot I_{p-1})(B_1 \cdot \ldots \cdot B_p) = \textstyle\sum_{s=1}^{p} B_1 \cdot \ldots \cdot AB_s \cdot \ldots \cdot B_p,$$

and the result follows from the observation that $(\alpha A)\mathbf{C}$ and $(\alpha A)(\mathbf{B} \cdot \mathbf{C})$ are similar sums of shuffle products.

3.5 LEMMA. *For any* $A \in \operatorname{End} V$ *and any one-generated* $\mathbf{C} \in \Pi_r \operatorname{End} \wedge^r V$ *one has* $A \times \mathbf{C} = A \cdot \mathbf{C} + (\alpha A)\mathbf{C}$.

Proof. Suppose that $\mathbf{C} \in \operatorname{End} \wedge^q V$, and use subscripts r to identify components of $\operatorname{End} \wedge^r V$. Then Lemma 3.4 yields

$$\begin{aligned}(\alpha A)(\alpha \mathbf{C})_r &= (\alpha A)(I_{r-q} \cdot \mathbf{C}) = (\alpha A)I_{r-q} \cdot \mathbf{C} + I_{r-q} \cdot (\alpha A)\mathbf{C} \\ &= (A \cdot I_{r-q-1}) \cdot \mathbf{C} + I_{r-q} \cdot (\alpha A)\mathbf{C} \\ &= I_{r-q-1} \cdot (A \cdot \mathbf{C}) + I_{r-q} \cdot (\alpha A)\mathbf{C} = \alpha(A \cdot \mathbf{C})_r + \alpha((\alpha A)\mathbf{C})_r,\end{aligned}$$

hence

$$(\alpha A)(\alpha \mathbf{C}) = \alpha(A \cdot \mathbf{C} + (\alpha A)\mathbf{C}) \in \Pi_r \operatorname{End} \wedge^r V,$$

hence

$$A \times \mathbf{C} = \alpha^{-1}((\alpha A)(\alpha \mathbf{C})) = A \cdot \mathbf{C} + (\alpha A)\mathbf{C}$$

as claimed.

3.6 LEMMA. *For any one-generated elements* $\mathbf{A} = A_p \in \operatorname{End} \wedge^p V$ *and* $\mathbf{B} = B_q \in \operatorname{End} \wedge^q V$ *one has* $(A_p \times B_q)_r = 0 \in \operatorname{End} \wedge^r V$ *for* $r > p + q$ *and* $(A_p \times B_q)_{p+q} = A_p \cdot B_q \in \operatorname{End} \wedge^{p+q} V$.

Proof. Let $J_{p+q-1} \subset \Pi_r \operatorname{End} \wedge^r V$ be the R-submodule consisting of the summands $\operatorname{End} \wedge^r V$ for $r < p + q$. It suffices to show by induction on p that $A_p \times B_q - A_p \cdot B_q \in J_{p+q-1}$, the case $p = 0$ being trivial. One may as well assume that $A_p = A_1 \cdot A_{p-1}$ for $A_1 \in \operatorname{End} V$ and a one-generated element $A_{p-1} \in \operatorname{End} \wedge^{p-1} V$. Then

$$A_p \times B_q = (A_1 \cdot A_{p-1}) \times B_q = (A_1 \times A_{p-1}) \times B_q - (\alpha A_1)A_{p-1} \times B_q$$

by Lemma 3.5, where $(\alpha A_1)A_{p-1} \times B_q \in J_{p+q-1}$ by a weak form of the inductive hypothesis. One also has $A_{p-1} \times B_q \in J_{p+q-1}$ by the same weak form of the inductive hypothesis, so that $(\alpha A_1)(A_{p-1} \times B_q) \in J_{p+q-1}$; hence a second application of Lemma 3.5 gives

$$A_p \times B_q = (A_1 \times A_{p-1}) \times B_q \bmod J_{p+q-1} = A_1 \times (A_{p-1} \times B_q) \bmod J_{p+q-1}$$
$$= A_1 \cdot (A_{p-1} \times B_q) \bmod J_{p+q-1} .$$

Finally, the specific form $A_{p-1} \times B_q - A_{p-1} \cdot B_q \in J_{p+q-2} (\subset J_{p+q-1})$ of the inductive hypothesis permits one to conclude that

$$A_p \times B_q = A_1 \cdot (A_{p-1} \cdot B_q) \bmod J_{p+q-1} = (A_1 \cdot A_{p-1}) \cdot B_q \bmod J_{p+q-1}$$
$$= A_p \cdot B_q \bmod J_{p+q-1} ,$$

which completes the inductive step.

3.7 Proposition. *For any module V over a commutative ring R with unit, the third product in $\Pi_r \operatorname{End} \wedge^r V$ restricts to a product in the submodule of one-generated elements of the direct sum $\amalg_r \operatorname{End} \wedge^r V$.*

Proof. Immediate consequence of Lemma 3.6.

Lemma 3.6 and Proposition 3.7 are certainly valid under considerably weaker hypotheses; for example, one can easily combine the present versions with localization techniques to obtain greater generality. One can possibly establish entirely unrestricted versions of Lemma 3.6 and Proposition 3.7 by applying the identity of Lemma 3.1 directly to elements

$$x_1 \wedge \dots \wedge x_r \in \wedge^r V \quad \text{for} \quad r \geqslant p + q ;$$

however, such a computation would probably be very complicated.

Here is a simple application of the results of this section. Any endomorphisms $A \in \operatorname{End} V$ and $B \in \operatorname{End} V$ are trivially one-generated, so that Lemmas 3.2 and 3.6 imply $A \times B = AB + A \cdot B$ and $B \times A = BA + B \cdot A$; since $A \cdot B = B \cdot A$ it follows that $AB - BA = A \times B - B \times A$. If V is traceable Theorem 2.3 then implies the best-known elementary property of the trace:

$$\operatorname{tr} AB - \operatorname{tr} BA = (\operatorname{tr} A)(\operatorname{tr} B) - (\operatorname{tr} B)(\operatorname{tr} A) = 0 .$$

4. Newton identities

Let A be any endomorphism of a module V over a commutative ring R with unit. The shuffle products of the compositions $I, A, A^2, \dots$ of A

generate a subalgebra R_A of one-generated elements of Π_r End $\wedge^r V$, and since the third product is defined in terms of shuffle products and composition products one expects various identities relating the three products in R_A. If V happens to be traceable one can then apply the trace and use Theorem 2.3 to obtain relations among the various trace-induced invariants of A, as indicated in the Introduction. In this section we develop those identities in R_A whose images under the trace are the Newton identities, relating the sums-of-powers invariants of A to the elementary invariants of A.

For convenience we henceforth impose an additional condition on the ground ring R itself: R will be a commutative algebra with unit over the field of rational numbers.

4.1 LEMMA. *For any $p \geqslant 0$ and any $q > 0$ the p-fold composition $A^p \in \mathrm{End}\, V$ and the q-fold shuffle product $\frac{1}{q!} A^{\cdot q} \in \mathrm{End} \wedge^q V$ satisfy*

$$(\alpha A^p)\left(\frac{1}{q!} A^{\cdot q}\right) = A^{p+1} \cdot \frac{1}{(q-1)!} A^{\cdot (q-1)}.$$

Proof. As in the proof of Lemma 3.4 one has

$$(\alpha A^p)(A^{\cdot q}) = (A^p \cdot I_{q-1})(A \cdot \ldots \cdot A) = \textstyle\sum_{s=1}^{q} A \cdot \ldots \cdot A^p A \cdot \ldots \cdot A$$
$$= q\, A^{p+1} \cdot A^{\cdot (q-1)},$$

the s^{th} summand containing $A^p A$ $(= A^{p+1})$ as its s^{th} factor, $s = 1, \ldots, q$.

4.2 LEMMA. *Let I_1 be the identity endomorphism in* End V; *then for any $r > 0$ and any $C_r \in \mathrm{End} \wedge^r V$ one has*

$$I_1 \times C_r = I_1 \cdot C_r + r\, C_r \in \mathrm{End} \wedge^{r+1} V \oplus \mathrm{End} \wedge^r V.$$

Proof. By Lemma 3.5 one has $I_1 \times C_r = I_1 \cdot C_r + (\alpha I_1) C_r$, where the degree r component of αI_1 is given by $I_{r-1} \cdot I_1 = r\, I_r \in \mathrm{End} \wedge^r V$.

4.3 THEOREM. *Let R be a commutative algebra with unit over the rational numbers, let V be any R-module, and let $A \in \mathrm{End}\, V$. Then the p-fold compositions $A^p \in \mathrm{End}\, V$ and the q-fold shuffle products*

$$\frac{1}{q!} A^{\cdot q} \in \mathrm{End} \wedge^q V$$

are related for each $r > 0$ by the identity

$$r\left(\frac{1}{r!}A^{\cdot r}\right) + \sum\nolimits_{p=1}^{r} (-1)^p A^p \times \frac{1}{q!}A^{\cdot q} = 0\,,$$

where $q = r - p$.

Proof. One applies Lemmas 3.5 and 4.1 to find

$$\begin{aligned} A^p \times \frac{1}{q!}A^{\cdot q} &= A^p \cdot \frac{1}{q!}A^{\cdot q} + (\alpha A^p)\left(\frac{1}{q!}A^{\cdot q}\right) \\ &= A^p \cdot \frac{1}{q!}A^{\cdot q} + A^{p+1} \cdot \frac{1}{(q-1)!}A^{\cdot(q-1)} \end{aligned}$$

for $q > 0$. Hence all summands of the alternating sum $\sum_{p+q=r} (-1)^p A^p \times \frac{1}{q!}A^{\cdot q}$ cancel except in the extreme cases $p = 0$ and $q = 0$. For $p = 0$ one uses Lemma 4.2 to find

$$A^0 \times \frac{1}{r!}A^{\cdot r} = A^0 \cdot \frac{1}{r!}A^{\cdot r} + r\left(\frac{1}{r!}A^{\cdot r}\right),$$

leaving the uncancelled term $r\left(\frac{1}{r!}A^{\cdot r}\right)$; for $q = 0$ one has $A^r \times \frac{1}{0!}A^{\cdot 0} = A^r \times I_0 = A^r$, for the third product unit element I_0, so that there are no further uncancelled terms.

Theorem 4.3 provides identities in the subalgebra $R_A \subset \Pi_r \operatorname{End} \wedge^r V$ generated by $A \in \operatorname{End} V$. In the next result $R_A[[t]]$ will be the formal power series algebra in a single indeterminate t over R_A. Observe that the element $I + tA = I_1 + tA \in R_A[[t]]$ has a composition-product inverse

$$(I+tA)^{-1} = I - tA + t^2A^2 - \ldots \in R_A[[t]]\,,$$

and that formal integration of $A(I+tA)^{-1}$ (from 0 to t) provides a composition-product logarithm

$$\ln(I+tA) = tA - \frac{t^2}{2}A^2 + \frac{t^3}{3}A^3 - \ldots \in R_A[[t]]$$

of $I + tA$. Similarly the shuffle-product exponentials

$$e^{\cdot tA} = I_0 + \frac{t}{1!}A + \frac{t^2}{2!}A \cdot A + \ldots \in R_A[[t]] \quad \text{and} \quad e^{\cdot (I+tA)^{-1}} \in R_A[[t]]$$

satisfy a composition-product identity

$$(\alpha e^{\cdot tA})e^{\cdot (I+tA)^{-1}} = (e^{\cdot (I+tA)})\,(e^{\cdot (I+tA)^{-1}}) = \mathbf{I} = e^{I};$$

hence

$$e^{\cdot tA} \times \alpha^{-1}e^{\cdot (I+tA)^{-1}} = \alpha^{-1}((\alpha e^{\cdot tA})\,(e^{\cdot (I+tA)^{-1}}) = \alpha^{-1}e^{I} = I_0$$

and similarly $\alpha^{-1}e^{\cdot (I+tA)^{-1}} \times e^{\cdot tA} = I_0$. Thus $\alpha^{-1}e^{\cdot (I+tA)^{-1}} \in R_A[[t]]$ is a two-sided third-product inverse $(e^{\cdot tA})^{-1}$ of $e^{\cdot tA}$, so that formal integration of the third-product $(A \cdot e^{\cdot tA}) \times (e^{\cdot tA})^{-1} \in R_A[[t]]$ (from 0 to t) provides a third-product logarithm $\ln^{\times}(e^{\cdot tA}) \in R_A[[t]]$ of the shuffle-product exponential $e^{\cdot tA} \in R_A[[t]]$.

4.4 THEOREM. *Let R be a commutative algebra with unit over the rational numbers, let V be any R-module, and let $A \in \mathrm{End}\, V$. Then the p-fold compositions $A^p \in \mathrm{End}\, V$ and the q-fold shuffle products*

$\frac{1}{q!}A^{\cdot q} \in \mathrm{End} \wedge^q V$ *are related in the formal power series ring $R_A[[t]]$ by the identity*

$$\ln (I+tA) = \ln^{\times}(e^{\cdot tA}).$$

Proof. For each $r > 0$ one can re-write Theorem 4.3 in the form

$$A\left(\sum_{p=0}^{r-1}(-1)^p A^p\right) \times \frac{1}{q!}A^{\cdot q} = A \cdot \frac{1}{(r-1)!}A^{\cdot (r-1)},$$

where $p + q = r - 1$. If one multiplies each such identity by t^{r-1} and computes formal sums over all values $0, 1, 2, \ldots$ of $r - 1$, the result is a formal power series identity

$$A(I+tA)^{-1} \times e^{\cdot tA} = A \cdot e^{\cdot tA} \in R_A[[t]].$$

Since $e^{\cdot tA}$ has a third-product inverse $(e^{\cdot tA})^{-1} \in R_A[[t]]$ it follows that

$$A(I+tA)^{-1} = (A \cdot e^{\cdot tA}) \times (e^{\cdot tA})^{-1};$$

that is,

$$\frac{d}{dt}\ln (I+tA) = \frac{d}{dt}\ln^{\times}(e^{\cdot tA}) \in R_A[[t]].$$

It remains only to integrate each power of t separately (from 0 to t) to complete the proof.

4.5 *Definition.* Let R be a commutative algebra with unit over the rational numbers, let V be a traceable R-module of rank n, and let $A \in \mathrm{End}\, V$.

Then the *elementary invariants* $\sigma_1, \sigma_2, \ldots, \sigma_n \in R$ of A and the *sums-of-powers invariants* $s_1, s_2, \ldots \in R$ of A are given by $\sigma_q = \sigma_q(A) = \operatorname{tr} \frac{1}{q!} A^{\cdot q}$ and $s_p = s_p(A) = \operatorname{tr} A^p$, respectively. Observe that since V is traceable of rank n one automatically has $A^{\cdot q} = 0$ hence $\sigma_q = 0$ whenever $q > n$; consequently the preceding hypotheses imply that

$$\operatorname{tr} e^{\cdot tA} = 1 + t\sigma_1 + \ldots + t^n \sigma_n \in R[[t]] .$$

4.6 Corollary (The Newton identities, more or less). *Let A be any endomorphism of a traceable module V of rank $n > 0$ over a commutative algebra R with unit over the rational numbers. Then for each $r > 0$ the invariants $\sigma_q = \sigma_q(A) \in R$ and $s_q = s_q(A) \in R$ satisfy the identity*

$$r\,\sigma_r + \textstyle\sum_{p=1}^{r} (-1)^p s_p\, \sigma_{r-p} = 0 ,$$

where $\sigma_q = 0$ for $q > n$.

Proof. By Theorem 2.3 the trace induces an algebra homomorphism $R_A \to R$ from the subalgebra R_A of the third-product algebra $\Pi_s \operatorname{End} \wedge^s V$ to the ground ring R; hence it suffices to apply the trace to the identities of Theorem 4.3.

4.7 Corollary. *Let A be any endomorphism of a traceable module V of rank $n > 0$ over a commutative algebra R with unit over the rational numbers. Then the elementary invariants $\sigma_1, \ldots, \sigma_n$ and sums-of-powers invariants $s_1, s_2, \ldots$ of A satisfy the identity*

$$t\,s_1 - \frac{t^2}{2} s_2 + \frac{t^3}{3} s_3 - \ldots = \ln(1 + t\sigma_1 + \ldots + t^n\sigma_n)$$

in the formal power series ring $R[[t]]$.

Proof. As in the preceding proof Theorem 2.3 implies that the trace induces an algebra homomorphism $R_A[[t]] \to R[[t]]$, products in $R_A[[t]]$ being defined in terms of third-products; hence it suffices to apply the trace to the identity of Theorem 4.4.

A special case of the third product and Theorem 2.3 appear briefly at the end of [2], somewhat fettered by the details of a specific application. The present article shows that the third product is a reasonable general feature of elementary linear algebra over any commutative ring with unit: it turns the trace into an algebra homomorphism, and it provides a natural setting for the study of invariants of module endomorphisms.

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