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XII. The Closed Graph Theorem

Let $\mathfrak{E}$, $\mathfrak{F}$ be definite spaces in the sense of Definition 15 over a field k whose valuation topology satisfies the 1. axiom of countability. For $f:\mathfrak{E} \to \mathfrak{F}$ a linear map set $\mathfrak{G}(f) := \{(\mathfrak{x}, \mathfrak{y}) \in \mathfrak{E} \oplus \mathfrak{F} \mid \mathfrak{y} = f(\mathfrak{x})\}$. Then [21] the "closed graph theorem" can be proved by classical methods (Baire category arguments):

$$\mathfrak{G}(f) \text{ is closed} \Rightarrow f \text{ is continuous}. \tag{24}$$

There is the following algebraic analogue of statement (24):

$$\mathfrak{G}(f) = \mathfrak{G}(f)^{\perp\perp} \Rightarrow f \text{ is } \perp\text{-continuous} \tag{25}$$

Here $\mathfrak{G}(f)^{\perp\perp}$ is taken in $\mathfrak{E} \overset{\perp}{\oplus} \mathfrak{F}$ and, by definition, f is $\perp$-continuous iff f is continuous with respect to the topologies on $\mathfrak{E}$ and $\mathfrak{F}$ whose 0-neighbourhood filters are generated by the orthogonals of all *finite* dimensional subspaces of $\mathfrak{E}$ and $\mathfrak{F}$ respectively. For $\mathfrak{E}$ an orthomodular space implication (25) holds: $\mathfrak{G}(f) = \mathfrak{G}(f)^{\perp\perp}$ implies that $\mathfrak{G}(f)$ is closed since the form is continuous on $\mathfrak{E} \overset{\perp}{\oplus} \mathfrak{F}$; so f is continuous by (24). Further, if $\mathfrak{G} \subset \mathfrak{F}$ is the orthogonal of a finite dimensional subspace then $f^{-1}(\mathfrak{G})$ is closed, hence $f^{-1}(\mathfrak{G}) = (f^{-1}(\mathfrak{G}))^{\perp\perp}$ as $\mathfrak{E}$ is orthomodular. But $(f^{-1}(\mathfrak{G}))^{\perp}$ is finite dimensional, hence f is $\perp$-continuous.

In [31] nice examples of $f:\mathfrak{E} \to \mathfrak{F}$ are given which illustrate that (25) is in general violated.

XIII. A few open problems

All orthomodular spaces are meant to be infinite dimensional and different from the classical ones over **R**, **C**, **H**.

Problem 1. Are cardinalities of maximal orthogonal families in an orthomodular space always equal? The answer is "yes" for those in $\mathscr{E}$.

Problem 2. Give an example of an orthomodular space that contains an uncountable orthogonal family of non-zero vectors.

Problem 3. Does the implication

$$\mathfrak{A} + \mathfrak{B} = (\mathfrak{A}+\mathfrak{B})^{\perp\perp} \Rightarrow \mathfrak{A}^{\perp} + \mathfrak{B}^{\perp} = (\mathfrak{A}\cap\mathfrak{B})^{\perp}$$

hold for all pairs of $\perp$-closed subspaces $\mathfrak{A} = \mathfrak{A}^{\perp\perp}$, $\mathfrak{B} = \mathfrak{B}^{\perp\perp}$ in an orthomodular space? The answer is "yes" for orthomodular spaces in $\mathscr{E}$. Cf. Remark 3 in [31]. More generally, are there other elementary lattice theoretic statements (in the sense of first order logic) that are valid in all $L_{\perp\perp}(E)$ where $\mathfrak{E}$ is orthomodular?

Problem 4. Are there spaces $\mathfrak{E}$ in $\mathscr{D}$, $\mathscr{E}$ with $L_s(\mathfrak{E}) = L_{\perp\perp}(\mathfrak{E}) \underset{\neq}{\subset} L_c(\mathfrak{E})$?

Problem 5. An orthomodular space $\mathfrak{E}$ in $\mathscr{E}$ is *never* isometric to any of its proper subspaces $\mathfrak{X}$, although it does happen that $\mathfrak{E}$ is similar to a proper subspace $\mathfrak{X}$. However, Keller's space is not similar to any of its proper subspaces. Give an intrinsic description of the phenomenon. (See [21].)

Problem 6. Answer Keller's question in § 3 of the introduction: When is $\{A\}'$ commutative for selfadjoint A in the algebra $\mathscr{B}(\mathfrak{H})$ of bounded operators $\mathfrak{H} \to \mathfrak{H}$?

Problem 7. Let $\mathfrak{E}$ be an orthomodular space in $\mathscr{D}$ or $\mathscr{E}$ such that the types of the members of a maximal orthogonal family are all different. Let Λ be the (countable) set of these types. For each choice of a family $(\lambda_i)_{i\in\Lambda}$ of nonnegative real numbers with $\sum_\Lambda \lambda_i = 1$ there is a probability distribution $f: L_{\perp\perp}(\mathfrak{E}) \to [0, 1] \subset \mathbf{R}$ uniquely defined as follows: for $\mathfrak{X} \in L_{\perp\perp}(\mathfrak{E})$ set $f(\mathfrak{X}) := \sum_{i\in J} \lambda_i$ where the subset $J \subseteq \Lambda$ consists of the types of the members of any orthogonal basis of $\mathfrak{X}$. We have $f(\mathfrak{E}) = 1$, $f(0) = 0$, $f(\sum \mathfrak{X}_i) = \sum f(\mathfrak{X}_i)$ for any countable family $\mathfrak{X}_0, \mathfrak{X}_1, \ldots$ of mutually orthogonal ($\perp$-closed) subspaces. These are by no means all probability distributions on $\mathfrak{E}$. There is a host of other possibilities. Can one bring some order into this multitude?

Problem 8. Classify the definite spaces with admissible topology over fixed base field.

Problem 9. Study the orthogonal group of definite orthomodular spaces.