

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 31 (1985)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: ON A CLASS OF ORTHOMODULAR QUADRATIC SPACES
Autor: Gross, Herbert / Künzi, Urs-Martin
Kapitel: XIII. A FEW OPEN PROBLEMS
DOI: <https://doi.org/10.5169/seals-54565>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 02.09.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

XII. THE CLOSED GRAPH THEOREM

Let \mathfrak{E} , \mathfrak{F} be definite spaces in the sense of Definition 15 over a field k whose valuation topology satisfies the 1. axiom of countability. For $f: \mathfrak{E} \rightarrow \mathfrak{F}$ a linear map set $\mathfrak{G}(f) := \{(x, y) \in \mathfrak{E} \oplus \mathfrak{F} \mid y = f(x)\}$. Then [21] the “closed graph theorem” can be proved by classical methods (Baire category arguments):

$$(24) \quad \mathfrak{G}(f) \text{ is closed} \Rightarrow f \text{ is continuous.}$$

There is the following algebraic analogue of statement (24):

$$(25) \quad \mathfrak{G}(f) = \mathfrak{G}(f)^{\perp\perp} \Rightarrow f \text{ is } \perp\text{-continuous}$$

Here $\mathfrak{G}(f)^{\perp\perp}$ is taken in $\mathfrak{E} \oplus^{\perp} \mathfrak{F}$ and, by definition, f is \perp -continuous iff f is continuous with respect to the topologies on \mathfrak{E} and \mathfrak{F} whose 0-neighbourhood filters are generated by the orthogonals of all *finite* dimensional subspaces of \mathfrak{E} and \mathfrak{F} respectively. For \mathfrak{E} an orthomodular space implication (25) holds: $\mathfrak{G}(f) = \mathfrak{G}(f)^{\perp\perp}$ implies that $\mathfrak{G}(f)$ is closed since the form is continuous on $\mathfrak{E} \oplus^{\perp} \mathfrak{F}$; so f is continuous by (24). Further, if $\mathfrak{G} \subset \mathfrak{F}$ is the orthogonal of a finite dimensional subspace then $f^{-1}(\mathfrak{G})$ is closed, hence $f^{-1}(\mathfrak{G}) = (f^{-1}(\mathfrak{G}))^{\perp\perp}$ as \mathfrak{E} is orthomodular. But $(f^{-1}(\mathfrak{G}))^{\perp}$ is finite dimensional, hence f is \perp -continuous.

In [31] nice examples of $f: \mathfrak{E} \rightarrow \mathfrak{F}$ are given which illustrate that (25) is in general violated.

XIII. A FEW OPEN PROBLEMS

All orthomodular spaces are meant to be infinite dimensional and different from the classical ones over \mathbf{R} , \mathbf{C} , \mathbf{H} .

Problem 1. Are cardinalities of maximal orthogonal families in an orthomodular space always equal? The answer is “yes” for those in \mathcal{E} .

Problem 2. Give an example of an orthomodular space that contains an uncountable orthogonal family of non-zero vectors.

Problem 3. Does the implication

$$\mathfrak{A} + \mathfrak{B} = (\mathfrak{A} + \mathfrak{B})^{\perp\perp} \Rightarrow \mathfrak{A}^{\perp} + \mathfrak{B}^{\perp} = (\mathfrak{A} \cap \mathfrak{B})^{\perp}$$

hold for all pairs of \perp -closed subspaces $\mathfrak{U} = \mathfrak{U}^{\perp\perp}$, $\mathfrak{B} = \mathfrak{B}^{\perp\perp}$ in an orthomodular space? The answer is "yes" for orthomodular spaces in \mathcal{E} . Cf. Remark 3 in [31]. More generally, are there other elementary lattice theoretic statements (in the sense of first order logic) that are valid in all $L_{\perp\perp}(E)$ where \mathfrak{E} is orthomodular?

Problem 4. Are there spaces \mathfrak{E} in \mathcal{D} , \mathcal{E} with $L_s(\mathfrak{E}) = L_{\perp\perp}(\mathfrak{E}) \subsetneq L_c(\mathfrak{E})$?

Problem 5. An orthomodular space \mathfrak{E} in \mathcal{E} is *never* isometric to any of its proper subspaces \mathfrak{X} , although it does happen that \mathfrak{E} is similar to a proper subspace \mathfrak{X} . However, Keller's space is not similar to any of its proper subspaces. Give an intrinsic description of the phenomenon. (See [21].)

Problem 6. Answer Keller's question in § 3 of the introduction: When is $\{A\}'$ commutative for selfadjoint A in the algebra $\mathcal{B}(\mathfrak{H})$ of bounded operators $\mathfrak{H} \rightarrow \mathfrak{H}$?

Problem 7. Let \mathfrak{E} be an orthomodular space in \mathcal{D} or \mathcal{E} such that the types of the members of a maximal orthogonal family are all different. Let Λ be the (countable) set of these types. For each choice of a family $(\lambda_i)_{i \in \Lambda}$ of nonnegative real numbers with $\sum_{\Lambda} \lambda_i = 1$ there is a probability distribution $f: L_{\perp\perp}(\mathfrak{E}) \rightarrow [0, 1] \subset \mathbf{R}$ uniquely defined as follows: for $\mathfrak{X} \in L_{\perp\perp}(\mathfrak{E})$ set $f(\mathfrak{X}) := \sum_{i \in J} \lambda_i$ where the subset $J \subseteq \Lambda$ consists of the types of the members of any orthogonal basis of \mathfrak{X} . We have $f(\mathfrak{E}) = 1$, $f(0) = 0$, $f(\sum \mathfrak{X}_i) = \sum f(\mathfrak{X}_i)$ for any countable family $\mathfrak{X}_0, \mathfrak{X}_1, \dots$ of mutually orthogonal (\perp -closed) subspaces. These are by no means all probability distributions on \mathfrak{E} . There is a host of other possibilities. Can one bring some order into this multitude?

Problem 8. Classify the definite spaces with admissible topology over fixed base field.

Problem 9. Study the orthogonal group of definite orthomodular spaces.