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X. CLIFFORD ALGEBRAS OF ORTHOMODULAR SPACES

X.1. ASSUMPTIONS. In Chap. X  $k$  is a commutative field of characteristic not 2 and  $\langle , \rangle$  is a symmetric bilinear anisotropic form  $\mathfrak{E} \times \mathfrak{E} \rightarrow k$  on the  $k$ -vector space  $\mathfrak{E}$ .

$C(\mathfrak{E})$  is the Clifford algebra of  $(\mathfrak{E}; \langle , \rangle)$ ; it is a  $k$ -algebra that contains the space  $\mathfrak{E}$  as a set of ring generators which satisfy  $x \cdot \eta + \eta \cdot x = 2\langle x, \eta \rangle$ . For any pair of elements  $c, d \in C(\mathfrak{E})$  there exists a finite orthogonal family  $e_0, \dots, e_n$  in  $\mathfrak{E}$  such that  $c = \sum_I \alpha_I e_I, d = \sum_I \beta_I e_I$ ; here the summation index  $I$  runs over all subsets

$I = \{i_1 < \dots < i_r\}$  of  $\{0, 1, \dots, n\}$  and  $e_I := e_{i_1} \cdot e_{i_2} \cdot \dots \cdot e_{i_r}$ ; the empty product  $e_\emptyset$  is the unit element in  $C(\mathfrak{E})$ .

There is a *canonical* symmetric bilinear form  $\langle , \rangle$  on  $C(\mathfrak{E})$  which extends the given form on  $\mathfrak{E}$  ([5, 11, 22]). One has

$$(16) \quad \langle c, d \rangle = \sum_I \alpha_I \beta_I \prod_{i \in I} \langle e_i, e_i \rangle$$

From now on we shall assume that  $(\mathfrak{E}; \langle , \rangle)$  is an infinite dimensional definite space.

X.2. CLIFFORD ALGEBRAS OF DEFINITE SPACES. In [6] Angela Fässler has proved that for certain definite orthomodular spaces  $\mathfrak{E}$  the algebra  $C(\mathfrak{E})$  is a skew field; furthermore, the  $k$ -vector space  $C(\mathfrak{E})$  equipped with the form (16) is a definite space whose completion  $\tilde{C}(\mathfrak{E})$  is orthomodular again. Furthermore  $\tilde{C}(\mathfrak{E})$  is a skew field, in fact, a  $*$ -valued field with  $*$  the extension to  $\tilde{C}(\mathfrak{E})$  of the main antiautomorphism of the Clifford algebra  $C(\mathfrak{E})$ ; the residue class field of  $\tilde{C}(\mathfrak{E})$  is isomorphic to the residue class field of  $\mathfrak{E}$ .

In the following theorem we prove the main fact in a simplified and slightly more general setting.

THEOREM 37. Assume that in the definite space  $(\mathfrak{E}; \langle , \rangle)$  each orthogonal family  $e_0, \dots, e_n$  has

$$(17) \quad \varphi\langle e_0 \rangle + \dots + \varphi\langle e_n \rangle \notin 2\Gamma$$

Then :

- (i)  $C(\mathfrak{E})$  equipped with the form in (16) is a definite space,
- (ii)  $C(\mathfrak{E})$  is a division ring,

(iii) The map  $\tilde{\varphi}: C(\mathfrak{E}) \rightarrow \Gamma \cup \{\infty\}$  defined by  $c \mapsto \varphi\langle c \rangle$  is a  $*$ -valuation for  $*$  the main antiautomorphism of  $C(\mathfrak{E})$ .

*Proof.* (i) It suffices to prove the triangle inequality (Lemma 14 (i)). Write  $c = \sum \alpha_I e_I$ ,  $d = \sum \beta_I e_I$  as in X.1. Then we have  $\varphi\langle \alpha e_I \rangle \neq \varphi\langle \beta e_J \rangle$  for  $I \neq J$  and  $\alpha \neq 0 \neq \beta$ . Hence

$$\varphi\langle c \rangle = \varphi\langle \sum \alpha_I e_I \rangle = \min_I \{\varphi\langle \alpha_I e_I \rangle\}$$

and similarly for  $\varphi\langle d \rangle$ . Therefore

$$\begin{aligned} \varphi\langle c+d \rangle &= \varphi\langle \sum (\alpha_I + \beta_I) e_I \rangle \geq \min \{2\varphi(\alpha_I + \beta_I) + \varphi\langle e_I \rangle\} \\ &\geq \min \{2\varphi\alpha_I + \varphi\langle e_I \rangle, 2\varphi\beta_I + \varphi\langle e_I \rangle\} = \min \{\varphi\langle c \rangle, \varphi\langle d \rangle\}. \end{aligned}$$

This proves (i). Next we show

$$(18) \quad \varphi\langle c \cdot d \rangle = \varphi\langle c \rangle + \varphi\langle d \rangle$$

Indeed, from

$$\langle e_I \cdot e_J \rangle = \langle \pm \langle e_{I \cap J} \rangle e_{(I \cup J) \setminus (I \cap J)} \rangle = \langle e_{I \cap J} \rangle^2 \langle e_{(I \cup J) \setminus (I \cap J)} \rangle = \langle e_I \rangle \cdot \langle e_J \rangle$$

we see that

$$\varphi\langle \alpha_I e_I \cdot e_J \rangle \leq \varphi\langle \alpha_I e_I \rangle \ \& \ \varphi\langle \beta_J e_J \cdot e_I \rangle \leq \varphi\langle \beta_J e_J \rangle$$

implies

$$\varphi\langle \alpha_I, \beta_J, e_I, e_J \rangle \leq \varphi\langle \alpha_I \beta_J e_I e_J \rangle.$$

We therefore pick  $G, H \subseteq \{0, \dots, n\}$  such that for all  $I \subset \{0, \dots, n\}$  we shall have

$$\varphi\langle \alpha_G e_G \rangle \leq \varphi\langle \alpha_I e_I \rangle, \varphi\langle \beta_H e_H \rangle \leq \varphi\langle \beta_I e_I \rangle.$$

It now follows that

$$\begin{aligned} \varphi\langle c \cdot d \rangle &= \varphi\langle (\sum \alpha_I e_I) \cdot \sum \beta_J e_J \rangle = \varphi\langle \sum \alpha_I \beta_J e_I e_J \rangle = \varphi\langle \alpha_G \beta_H e_G e_H \\ &\quad + \sum' \alpha_I \beta_J e_I e_J \rangle = \varphi\langle \alpha_G \beta_H e_G e_H \rangle = \varphi\langle c \rangle + \varphi\langle d \rangle. \end{aligned}$$

Thus (18) is established.

From (18) it follows that  $C(\mathfrak{E})$  has no zero divisors, hence  $C(\mathfrak{E})$  is a division ring (being an inductive limit of finite dimensional algebras). The map  $\tilde{\varphi}: C(\mathfrak{E}) \rightarrow \Gamma \cup \{\infty\}$  as defined in (iii) of the Theorem is a  $*$ -valuation, for  $\tilde{\varphi}(c^*) = \tilde{\varphi}(c)$  is obvious and everything else has been established already.

COROLLARY 38. Assume that the definite space  $(\mathfrak{E}; \langle , \rangle)$  is complete and that the system of types (Corollary 26) is linearly independent in  $\Gamma/2\Gamma$  (considered as a  $\mathbf{Z}_2$ -vector space) then the conclusions (i), (ii), (iii) of Theorem 37 hold.

$C(\mathfrak{E})$  in Theorem 37 is not complete (unless finite dimensional). Its quadratic form  $\langle , \rangle$  can be extended to the completion  $\tilde{C}$ . By using Theorem 28 one can see that this completion has  $L_{\perp\perp}(\tilde{C}) = L_c(\tilde{C})$  if and only if  $E$  has  $L_{\perp\perp}(E) = L_c(E)$ .

### XI. CONTINUOUS OPERATORS ARE NOT ALWAYS BOUNDED

XI.1. INTRODUCTION. Let  $\mathfrak{E}$  be an infinite dimensional definite space in the sense of Definition 15. A linear map (operator)  $h: \mathfrak{E} \rightarrow \mathfrak{E}$  is called *bounded* iff there exists  $\gamma \in \Gamma$  such that for all  $x \in \mathfrak{E}$  we have  $\varphi\langle hx \rangle \geq \gamma + \varphi\langle x \rangle$ .

In [6] A. Fässler gave an explicit example of a continuous operator  $h$  on an orthomodular space  $\mathfrak{E}$  that is not bounded; she also proved a criterion for boundness which is very useful in the study of the algebra  $\mathcal{B}(\mathfrak{E})$  of bounded operators  $h: \mathfrak{E} \rightarrow \mathfrak{E}$  when  $\mathfrak{E}$  is an orthomodular definite space of a certain kind. We shall prove this criterion anew here as its original proof can be shortened considerably.

We shall consider definite spaces that satisfy

(19)  $(\mathfrak{E}; \langle , \rangle)$  contains a maximal orthogonal family  $(e_i)_{\mathbf{N}}$  such that the groups  $\Theta(\varphi\langle e_i \rangle)$  are different.

By (14) we see that (19) is a property of  $\mathfrak{E}$ , not of  $(e_i)_{\mathbf{N}}$ ; Keller's original example of an orthomodular space satisfies (19).

XI.2. FÄSSLER'S CRITERION. In this subsection let  $(\mathfrak{E}; \langle , \rangle)$  be an infinite dimensional orthomodular space that has (19). Fix a maximal orthogonal family  $(e_i)_{\mathbf{N}}$  that enjoys (19). If  $f: \mathfrak{E} \rightarrow \mathfrak{E}$  is given, expand (Lemma 27)

$$(20) \quad f e_i = \sum_{j \in \mathbf{N}} \alpha_{ij} e_j \quad (i \in \mathbf{N})$$

THEOREM 39 ([6]). The linear map  $f$  is bounded iff it is continuous and satisfies