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Remark 30. By Theorem 28 the isometry type of a definite space with admissible topology is characterized by the sequence $(\langle \mathfrak{e}_i \rangle)_{i \in \mathbf{N}}$ where $(\mathfrak{e}_i)_{i \in \mathbf{N}}$ is a maximal orthogonal family in $\mathfrak{E}$. Conversely, for each $(\alpha_i) \in k^{\mathbf{N}}$ there is a definite space $\mathfrak{E}$ with $L_c(\mathfrak{E}) = L_s(\mathfrak{E})$ admitting a maximal orthogonal family $(\mathfrak{e}_i)_{i \in \mathbf{N}}$ with $\langle \mathfrak{e}_i \rangle = \alpha_i$ $(i \in \mathbf{N})$ provided that

(A) $\xi_i := \varphi \alpha_i \in \Gamma$ satisfies the (type-) condition expressed in (8)

(B) The form $\langle \; , \; \rangle$ defined on $\mathfrak{F} := k(\mathfrak{e}_i)_{i \in \mathbf{N}}$ by $\langle \mathfrak{e}_i, \mathfrak{e}_j \rangle = 0$ $(i \neq j)$, $\langle \mathfrak{e}_i \rangle = \alpha_i$ $(i \in \mathbf{N})$ is definite.

These two conditions are implemented by many fields. In order to satisfy (A) one may, e.g. pick fields of generalized formal power series that are complete under a valuation φ with group Γ a prescribed Hahn product [30, p. 31] with sufficiently many factors not 2-divisible, e.g. $\Gamma = \mathbf{Z}^{(\mathbf{N})}$ ordered antilexicographically. Let k be any field with (A) and $t \in \Gamma/2\Gamma$; set $\mathfrak{F}_t = \{\text{span } \mathfrak{e}_i \mid \varphi\alpha_i + 2\Gamma = t\}$. By (A) $\dim \mathfrak{F}_t < \infty$; furthermore

$$\mathfrak{F} = \oplus^{\perp} \{\mathfrak{F}_t \mid t \in \Gamma/2\Gamma\} .$$

In order to check whether the form $\langle \; , \; \rangle$ satisfies the triangle inequality on $\mathfrak{F}$ it suffices to verify said inequality on each $\mathfrak{F}_t$. A. Fässler has given a handy criterium for $\langle \; , \; \rangle$ to be definite if Hahnproducts Γ are used, as indicated, to construct k with (A), [6, Lemma 15, 16].

VIII. Appendix: Extending the Main Theorem to the class $\mathscr{E}$ of norm-topological spaces

The arguments applied to the spaces in the class $\mathscr{D}$ can be extended to a larger class $\mathscr{E}$. First we have (cf. Definition 15):

Definition 31. An infinite dimensional anisotropic quadratic space $(\mathfrak{E}; \langle \; , \; \rangle)$ over a *-valued field $(k, *, \varphi, \Gamma)$ is called norm-topological if the sets $\mathfrak{U}_\gamma := \{\mathfrak{x} \in \mathfrak{E} \mid \varphi\langle \mathfrak{x} \rangle > \gamma\}$ form a 0-neighbourhood basis of a vector space topology on $\mathfrak{E}$. Let $\mathscr{E}$ be the class of all norm-topological spaces.

Definite spaces are norm-topological, obviously.

A proper subgroup Δ of Γ is *convex* (or isolated) if "$0 \leqslant x \leqslant y$ & $y \in \Delta$" implies "$x \in \Delta$". If the subgroup $\Delta \subset \Gamma$ is convex then the factor group Γ/Δ is ordered by setting $\gamma + \Delta \leqslant \delta + \Delta$ iff $\gamma < \delta$ or $\gamma - \delta \in \Delta$; furthermore, $\varphi_\Delta: k \to \Gamma/\Delta \cup \{\infty\}$ defined by $\varphi_\Delta(\alpha) = \varphi(\alpha) + \Delta$ is a valuation (a "coarser valuation") which yields the same topology on k as φ.

In order to make the mechanism of types work in the context of norm-topological spaces, i.e., in order to salvage the statement of Corollary 26 in the new context, the concept of type has to be coarsened as follows. For $\gamma \in \Gamma$ we introduce

$$\Delta(\gamma) := \{\delta \in \Gamma \mid \forall n \in \mathbf{N} : n \mid \delta \mid \leqslant \mid \gamma \mid\} \tag{12}$$

and

$$\Theta(\gamma) := \bigcap_{\delta \in \Gamma} \Delta(\gamma + 2\delta) \tag{13}$$

If $\gamma \neq 0$ then $\Delta(\gamma)$ is the largest convex subgroup of Γ not containing γ ([21]).

Remark 32. The group defined in (13) for $\gamma = \varphi\langle \mathfrak{e} \rangle$, $\mathfrak{e} \in \mathfrak{E}$, represents yet another possibility to introduce a "type" for the vectors in a definite space. The fundamental property expressed in Lemma 25 can be replaced and reproved (along the same lines), cf. [21]:

(14) If U is a convex subgroup in Γ and $(\mathfrak{e}_i)_{\mathbf{N}}$, $(\mathfrak{f}_i)_{\mathbf{N}}$ are two maximal orthogonal families in a norm-topological space that satisfies (iii) in Theorem 28 then

$$\operatorname{card}\{i \in I \mid \Theta(\varphi\langle \mathfrak{e}_i \rangle \subsetneq U\} = \operatorname{card}\{j \in \mathbf{N} \mid \Theta(\varphi\langle \mathfrak{f}_j \rangle) \subsetneq U\}.$$

One has the following analogue of Lemma 14:

LEMMA 33. ([21]). *Let* $(\mathfrak{E}; \langle\ ,\ \rangle; \varphi, \Gamma, *)$ *be a norm-topological space and* $\varphi(2) = 0$ *(cf. Remark 35 below). Then there is a valuation* $\tilde{\varphi}: k \to \tilde{\Gamma} \cup \{\infty\}$ *coarser than* φ *such that the following holds: Either* $(\mathfrak{E}; \langle\ ,\ \rangle; \tilde{\varphi}, \tilde{\Gamma}, *)$ *is a definite space, in the sense of Definition 15, or else there are no analytically nilpotent elements* $\alpha \in k$ *(i.e., for no* $\alpha \neq 0$ *shall we have* $\lim_{\mathbf{N}} \alpha^n = 0$) *and then the following weakened versions of the statements in Lemma 14 hold:*

(i)′ $\tilde{\varphi}_\Delta\langle \mathfrak{x} + \mathfrak{y} \rangle \geqslant \min\{\tilde{\varphi}_\Delta\langle \mathfrak{x} \rangle, \tilde{\varphi}_\Delta\langle \mathfrak{y} \rangle\}$

(ii)′ $\tilde{\varphi}\langle \mathfrak{x} \rangle \leqslant \tilde{\varphi}\langle \mathfrak{y} \rangle \;\&\; \langle \mathfrak{x}, \mathfrak{y} \rangle = 0 \Rightarrow \tilde{\varphi}_\Lambda\langle \mathfrak{x} \rangle = \tilde{\varphi}_\Lambda\langle \mathfrak{x} + \mathfrak{y} \rangle$

(iii)′ $\tilde{\varphi}\langle \mathfrak{x}, \mathfrak{y} \rangle \geqslant \min\{\tilde{\varphi}_\Delta\langle \mathfrak{x} \rangle, \tilde{\varphi}_\Delta\langle \mathfrak{y} \rangle\}$

(iv)′ $2\tilde{\varphi}_\Delta\langle \mathfrak{x}, \mathfrak{y} \rangle \geqslant \tilde{\varphi}_\Delta\langle \mathfrak{x} \rangle + \tilde{\varphi}_\Delta\langle \mathfrak{y} \rangle$
where $\Lambda = \Theta(\tilde{\varphi}\langle \mathfrak{x} \rangle)$ *and* $\Delta = \Theta(\tilde{\varphi}\langle \mathfrak{x} \rangle) \cap \Theta(\tilde{\varphi}\langle \mathfrak{y} \rangle)$.

The inequalities in Lemma 33 suffice to salvage all results proved previously on definite spaces; in particular we have the following strengthening of Theorem 28 (cf. Remark 35 below):

THEOREM 34 [21]. *Let* $\mathfrak{E}$ *be a norm-topological space in the sense of Definition 31 and assume* $\varphi(2) = 0$. *Then the statements (i), (ii), (iii) in Theorem 28 are equivalent.*

Remark 35. In Definition 15, Lemma 33 and in Theorem 34 we stipulated that $\varphi(2) = 0$ for the valuation φ of the base field. However, it is neither necessary to assume this nor that char k be different from two. As technicalities increase if 2 is not a unit for φ the general case has been banned from this elementary survey. Refer to [21].

IX. APPENDIX: ORTHOMODULAR SPACES OVER ORDERED FIELDS

A Baer order of a *-field k is a subset $\Pi \subset S := \{\alpha \in k \mid \alpha = \alpha^*\}$ *with* $1 \in \Pi$, $0 \notin \Pi$, $\Pi + \Pi \subset \Pi$, $\forall \alpha \neq 0: \alpha\Pi\alpha^* \subset \Pi$, $-\Pi \,\dot{\cup}\, \Pi = S\backslash\{0\}$. ([14]). The map $\alpha \mapsto \alpha^*\alpha =: \|\alpha\|$ has the properties of a norm and defines a topology on k; if * is continuous then k is a topological *-field [14, Theorem 4.1, p. 231]. The theory of positive definite orthomodular spaces over archimedean ordered fields is settled in [9]: There are but the classical Hilbert spaces over **R**, **C**, **H**. If the order is non-archimedean we shall assume that

(15) the subgroup S generated by all $\alpha^*\alpha^{-1}$ is bounded.

There is [14, Sec. 4.5, p. 234] a valuation on k that induces the norm-topology. We remark that the boundness condition on S is always satisfied for the usual orderings on commutative fields, for Prestel's semi-orderings and for all *-ordered fields that are known hitherto.

A family $(e_\iota)_{\iota\in I}$ of vectors in a *positive definite space* $(\mathfrak{E}; \langle\ ,\ \rangle)$ over an ordered *-field k is said to satisfy the *type condition* (cf. Definition 21) iff for all $(\alpha_\iota)_{\iota\in I} \in k^I$ the following holds: if $(\langle\alpha_\iota e_\iota\rangle)_{\iota\in I}$ is bounded then $(\alpha_\iota e_\iota)_{\iota\in I}$ converges to $0 \in \mathfrak{E}$.

With this version of type condition we have

THEOREM 36. *Let* $(\mathfrak{E}; \langle\ ,\ \rangle)$ *be a positive definite space over a non-archimedean ordered *-field that satisfies (15). Then the statements (i), (ii), (iii) in Theorem 28 are equivalent.*