VII. The Main Theorem

We are now able to characterize the definite spaces whose topology is admissible (Def. 1). Refer to Definition 21 for "type condition".

THEOREM 28 [20]. *Let* $\mathfrak{E}$ *be a definite space in the sense of Definition 15. The following conditions are equivalent*

(i) $L_c(\mathfrak{E}) = L_s(\mathfrak{E})$ (cf. (1), (2), (3))

(ii) $L_c(\mathfrak{E}) = L_{\perp\perp}(\mathfrak{E})$ (*"the topology is admissible", Def. 1*)

(iii) k *is complete and* $\mathfrak{E}$ *is the completion of a* $\aleph_0$*-dimensional space spanned by an orthogonal basis that satisfies the type condition.*

Proof. (i) $\Rightarrow$ (ii) holds trivially because $L_s \subseteq L_{\perp\perp} \subseteq L_c$ by continuity of the form; (ii) $\Rightarrow$ (iii) was carried out in Chapter V. Just as in [18] we can establish (iii) $\Rightarrow$ (i). Let $\mathfrak{U} \in L_c(\mathfrak{E})$. Pick a maximal orthogonal family $(\mathfrak{v}_i)_{i\in I}$ in $\mathfrak{U}$ and extend it to a maximal orthogonal family $(\mathfrak{v}_i)_{I\cup J}$ in $\mathfrak{E}$. For $\mathfrak{x} \in \mathfrak{E}$ we have by Lemma 27 $\mathfrak{x} = \mathfrak{x}' + \mathfrak{x}''$ where $\mathfrak{x}' = \sum_I \langle \mathfrak{x}, \mathfrak{v}_i\rangle \langle \mathfrak{v}_i\rangle^{-1}\mathfrak{v}_i$ and $\mathfrak{x}'' = \sum_J \langle \mathfrak{x}, \mathfrak{v}_i\rangle \langle \mathfrak{v}_i\rangle^{-1}\mathfrak{v}_i$. Now $\mathfrak{x}' \in \overline{\mathfrak{U}} = \mathfrak{U}$. All that remains to be shown is $\mathfrak{x}'' \in \mathfrak{U}^\perp$. Now $\mathfrak{U}^\perp$ is closed so it suffices to show that $\mathfrak{v}_i \in \mathfrak{U}^\perp$ for all $i \in J$. To this end pick $\mathfrak{u} \in \mathfrak{U}$ and decompose $\mathfrak{u} = \mathfrak{u}' + \mathfrak{u}''$ (analogous to the decomposition of $\mathfrak{x}$): $\mathfrak{u}'' = \mathfrak{u} - \mathfrak{u}' \in \mathfrak{U} - \mathfrak{U} = \mathfrak{U}$. Now $\langle \mathfrak{u}'', \mathfrak{v}_i\rangle = 0$ for all $i \in I$ so $\mathfrak{u}'' = 0$ since $(\mathfrak{v}_i)_{i\in I}$ is a maximal orthogonal family. From

$$0 = \mathfrak{u}'' := \sum_J \langle \mathfrak{u}, \mathfrak{v}_i\rangle \langle \mathfrak{v}_i\rangle^{-1}\mathfrak{v}_i$$

we obtain $\langle \mathfrak{u}, \mathfrak{v}_i\rangle = 0$ $(i\in J)$. As $\mathfrak{u} \in \mathfrak{U}$ was arbitrary this says that $\mathfrak{v}_i \in \mathfrak{U}^\perp$ $(i\in J)$.

Q.E.D.

Remark 29. Let the definite space $\mathfrak{E}$ be the completion of $\mathfrak{F} = k(\mathfrak{e}_i)_{i\in\mathbf{N}}$, $(\mathfrak{e}_i)_{\mathbf{N}}$ an orthogonal family (that does not necessarily satisfy the type condition). If k is complete then $\mathfrak{E}$ is isometric to the k-space $\hat{\mathfrak{F}}$ of all sequences $(\lambda_i)_{i\in\mathbf{N}} \in k^{\mathbf{N}}$ such that $\lim_{\mathbf{N}}(2\varphi\lambda_i + \varphi\langle \mathfrak{e}_i\rangle) = \infty$ and equipped with the form $\langle(\lambda_i), (\mu_i)\rangle = \sum_{\mathbf{N}} \lambda_i\mu_i\langle \mathfrak{e}_i\rangle$. Indeed, the set $\hat{\mathfrak{F}}$ is a definite k-space and the map $\Psi: (\lambda_i) \to \sum \lambda_i e_i$ is a well defined isometry $\hat{\mathfrak{F}} \to \Psi(\hat{\mathfrak{F}}) \subset \mathfrak{E}$. By the "infinite Pythagoras" we have $\ker \Psi = 0$; on the other hand, Lemma 16 shows that Ψ is also surjective.

Thus *all* definite spaces that carry an admissible topology are (by Theorem 28) of the kind invented by Keller.

Remark 30. By Theorem 28 the isometry type of a definite space with admissible topology is characterized by the sequence $(\langle \mathfrak{e}_i \rangle)_{i \in \mathbf{N}}$ where $(\mathfrak{e}_i)_{i \in \mathbf{N}}$ is a maximal orthogonal family in $\mathfrak{E}$. Conversely, for each $(\alpha_i) \in k^{\mathbf{N}}$ there is a definite space $\mathfrak{E}$ with $L_c(\mathfrak{E}) = L_s(\mathfrak{E})$ admitting a maximal orthogonal family $(\mathfrak{e}_i)_{i \in \mathbf{N}}$ with $\langle \mathfrak{e}_i \rangle = \alpha_i$ $(i \in \mathbf{N})$ provided that

(A) $\xi_i := \varphi\alpha_i \in \Gamma$ satisfies the (type-) condition expressed in (8)

(B) The form $\langle\ ,\ \rangle$ defined on $\mathfrak{F} := k(\mathfrak{e}_i)_{i \in \mathbf{N}}$ by $\langle \mathfrak{e}_i, \mathfrak{e}_j \rangle = 0$ $(i \neq j)$, $\langle \mathfrak{e}_i \rangle = \alpha_i$ $(i \in \mathbf{N})$ is definite.

These two conditions are implemented by many fields. In order to satisfy (A) one may, e.g. pick fields of generalized formal power series that are complete under a valuation φ with group Γ a prescribed Hahn product [30, p. 31] with sufficiently many factors not 2-divisible, e.g. $\Gamma = \mathbf{Z}^{(\mathbf{N})}$ ordered antilexicographically. Let k be any field with (A) and $t \in \Gamma/2\Gamma$; set $\mathfrak{F}_t = \{\text{span } \mathfrak{e}_i \mid \varphi\alpha_i + 2\Gamma = t\}$. By (A) $\dim \mathfrak{F}_t < \infty$; furthermore

$$\mathfrak{F} = \oplus^{\perp} \{\mathfrak{F}_t \mid t \in \Gamma/2\Gamma\}.$$

In order to check whether the form $\langle\ ,\ \rangle$ satisfies the triangle inequality on $\mathfrak{F}$ it suffices to verify said inequality on each $\mathfrak{F}_t$. A. Fässler has given a handy criterium for $\langle\ ,\ \rangle$ to be definite if Hahnproducts Γ are used, as indicated, to construct k with (A), [6, Lemma 15, 16].

VIII. Appendix: Extending the Main Theorem to the class $\mathscr{E}$ of norm-topological spaces

The arguments applied to the spaces in the class $\mathscr{D}$ can be extended to a larger class $\mathscr{E}$. First we have (cf. Definition 15):

Definition 31. An infinite dimensional anisotropic quadratic space $(\mathfrak{E}; \langle\ ,\ \rangle)$ over a $*$-valued field $(k, *, \varphi, \Gamma)$ is called norm-topological if the sets $\mathfrak{U}_\gamma := \{\mathfrak{x} \in \mathfrak{E} \mid \varphi\langle \mathfrak{x} \rangle > \gamma\}$ form a 0-neighbourhood basis of a vector space topology on $\mathfrak{E}$. Let $\mathscr{E}$ be the class of all norm-topological spaces. Definite spaces are norm-topological, obviously.

A proper subgroup Δ of Γ is *convex* (or isolated) if "$0 \leqslant x \leqslant y$ & $y \in \Delta$" implies "$x \in \Delta$". If the subgroup $\Delta \subset \Gamma$ is convex then the factor group Γ/Δ is ordered by setting $\gamma + \Delta \leqslant \delta + \Delta$ iff $\gamma < \delta$ or $\gamma - \delta \in \Delta$; furthermore, $\varphi_\Delta : k \to \Gamma/\Delta \cup \{\infty\}$ defined by $\varphi_\Delta(\alpha) = \varphi(\alpha) + \Delta$ is a valuation (a "coarser valuation") which yields the same topology on k as φ.