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straight lines in  $E$ . A family  $(e_i)_{i \in I}$  of vectors in  $\mathfrak{E}$  is said to satisfy the *type-condition* iff for all  $(\alpha_i)_{i \in I} \in k^I$  the following holds: if  $(\varphi \langle \alpha_i e_i \rangle)_{i \in I}$  is bounded (below) then  $(\alpha_i e_i)_{i \in I}$  converges to  $0 \in E$ .

**COROLLARY 22.** *Let  $\mathfrak{E}$  be as in Theorem 17.  $\Gamma/2\Gamma$  is infinite. Each orthogonal family in  $\mathfrak{E}$  satisfies the type-condition, equivalently,  $\Gamma/2\Gamma$  satisfies (8) below.*  $\square$

**COROLLARY 23.** *Let  $\mathfrak{E}$  be as in Theorem 17. Then  $k$  is complete.*

*Proof.* By Corollary 19 it suffices to show that a sequence  $(\alpha_i)_{i \in \mathbb{N}}$  with limit  $0 \in k$  is summable. Let  $(e_i)_{i \in \mathbb{N}}$  be maximal orthogonal in  $\mathfrak{E}$  with  $(\varphi \langle e_i \rangle)_{i \in \mathbb{N}}$  bounded below. If  $(\lambda_i)_{i \in \mathbb{N}} \in k^\mathbb{N}$  has  $(\varphi(\lambda_i))_{i \in \mathbb{N}}$  bounded below then  $(\lambda_i e_i)_{i \in \mathbb{N}}$  is summable and by continuity of  $\langle \cdot, \cdot \rangle$  we obtain

$$\left\langle \sum_{\mathbb{N}} \lambda_i e_i, \sum_{\mathbb{N}} e_i \right\rangle = \sum_{\mathbb{N}} \lambda_i \langle e_i \rangle.$$

Thus, all families  $(\lambda_i \langle e_i \rangle)_{i \in \mathbb{N}}$  with bounded  $(\lambda_i)_{i \in \mathbb{N}}$  are summable.

Pick a strictly monotonic sequence  $(n_i)_{i \in \mathbb{N}} \in \mathbb{N}^\mathbb{N}$  with  $n_0 = 0$  and for all  $i \in \mathbb{N}^+$  and all  $m \geq n_i$ :  $\varphi(\alpha_m) > \varphi \langle e_i \rangle$ , and set  $A_i := \sum \{\alpha_j \mid n_i \leq j < n_{i+1}\}$ . The family  $(A_i)_{i \in \mathbb{N}}$  is summable if and only if  $(\alpha_i)_{i \in \mathbb{N}}$  summable and, if the sums exist, these must be equal. If we set  $\lambda_i := A_i \langle e_i \rangle^{-1}$  then, by what we have shown, the family of the  $A_i = \lambda_i \langle e_i \rangle$  is summable.  $\square$

**COROLLARY 24.** *Let  $\mathfrak{E}$  be as in Theorem 17. Then  $\mathfrak{E}$  is complete.*

*Proof.* Let  $(x_i)_{i \in \mathbb{N}}$  be a Cauchy sequence (Corollary 19). For each fixed  $\eta \in \mathfrak{E}$  the map  $x \mapsto \langle \eta, x \rangle$  is uniformly continuous. Hence by Cor. 23 the map  $f: \eta \mapsto \lim_i \langle \eta, x_i \rangle$  is well-defined. As it is a continuous linear map, its kernel is a closed hyper-plane and so  $(L_c(\mathfrak{E}) = L_{\perp\perp}(\mathfrak{E}))$  there is  $a \in \mathfrak{E}$  such that  $f(\eta) = \langle \eta, a \rangle$ . Let  $N \subseteq \mathbb{N}$  be infinite. Because  $\lim_i \varphi \langle \eta, a - x_i \rangle = \infty$  for all  $\eta \in \mathfrak{E}$  it follows by systematic use of the Cauchy-Schwarz inequality that  $\{\varphi \langle a - x_i \rangle \mid i \in N\}$  is not bounded above by any  $\gamma \in \Gamma$ . Therefore  $(x_i)_{i \in \mathbb{N}}$  converges to  $a$ .

## VI. SUFFICIENT CONDITIONS IN $\mathcal{D}$ FOR $L_c = L_{\perp\perp}$

**VI.1. ASSUMPTIONS.** In this chapter  $(\mathfrak{E}; \langle \cdot, \cdot \rangle)$  is a definite space in the sense of Definition 15. Of the base field  $k$  we shall furthermore assume

(cf. Corollaries 22 and 23)

$\Gamma/2\Gamma$  contains a sequence  $(\xi_i + 2\Gamma)_{i \in \mathbb{N}}$  such that each

- (8) system of representatives  $(\xi_i + 2\gamma_i)_{i \in \mathbb{N}}$  that is bounded below tends to  $\infty$ .
- (9)  $k$  is complete.

Thus, by (8),  $\Gamma/2\Gamma$  will be infinite and the topology on  $k$  will satisfy the first countability axiom. There are many fields that satisfy (8) and (9): See Remark 30.

The results in the next sections will culminate in Theorem 28 which characterizes certain definite spaces that are orthomodular.

**VI.2. COUNTING TYPES.** Let  $\mathfrak{E}$  be the completion of an  $\aleph_0$ -dimensional space  $\mathfrak{F}$  which is spanned by an orthogonal basis  $\mathcal{B} = (\mathbf{e}_i)_{i \in \mathbb{N}}$  that satisfies the type condition (Def. 21).  $\mathfrak{F}$  is dense in  $\mathfrak{E}$  so  $\mathfrak{F}^\perp = \{0\}$  and hence  $\mathcal{B}$  is maximal. By Lemma 16 we have therefore  $\mathbf{x} = \sum_{\mathbb{N}} \langle \mathbf{x}, \mathbf{e}_i \rangle \langle \mathbf{e}_i \rangle^{-1} \mathbf{e}_i$  for all  $\mathbf{x} \in \mathfrak{E}$ .

We now introduce the function  $v$  which counts types on  $\mathcal{B}$ . Let  $v: \Gamma/2\Gamma \rightarrow \mathbb{N}: t \mapsto \text{card } \{i \in \mathbb{N} \mid T \circ \varphi \langle \mathbf{e}_i \rangle = t\}$  (cf. Def. 21). We have

**LEMMA 25.** *If  $\mathbf{f}_1, \dots, \mathbf{f}_m$  are pairwise orthogonal (non zero) vectors in  $\mathfrak{E}$  with  $T \circ \varphi \langle \mathbf{f}_i \rangle = t \in \Gamma/2\Gamma$  for all  $1 \leq i \leq m$  then  $m \leq v(t)$ .*

*Proof.* We shall replace the  $\mathbf{f}_i$  by suitable multiples and assume that  $\varphi \langle \mathbf{f}_i \rangle = \gamma \in \Gamma$  for all  $1 \leq i \leq m$ . Let  $J := \{i \in \mathbb{N} \mid T \circ \varphi \langle \mathbf{e}_i \rangle = t\}$ . We have  $\mathbf{f}_j = \mathbf{f}'_j + \mathbf{f}''_j$  where

$$\mathbf{f}'_j := \sum_J \langle \mathbf{f}_j, \mathbf{e}_i \rangle \langle \mathbf{e}_i \rangle^{-1} \mathbf{e}_i, \quad \mathbf{f}''_j := \sum_{\mathbb{N} \setminus J} \langle \mathbf{f}_j, \mathbf{e}_i \rangle \langle \mathbf{e}_i \rangle^{-1} \mathbf{e}_i.$$

Since Lemma 14 (ii) generalizes to finite as well as to infinite sums we find  $\varphi \langle \mathbf{f}''_j \rangle = \min_{i \in \mathbb{N} \setminus J} \{\varphi \langle \langle \mathbf{f}_j, \mathbf{e}_i \rangle \langle \mathbf{e}_i \rangle^{-1} \mathbf{e}_i \rangle\} \neq \varphi \langle \mathbf{f}_j \rangle$  (because types are different). By Lemma 14 (ii) furthermore  $\varphi \langle \mathbf{f}_j \rangle \leq \varphi \langle \mathbf{f}'_j \rangle$ ,  $\varphi \langle \mathbf{f}_j \rangle \leq \varphi \langle \mathbf{f}''_j \rangle$  and we must have equality in at least one instance. Therefore

$$(10) \quad \varphi \langle \mathbf{f}_j \rangle = \varphi \langle \mathbf{f}'_j \rangle = \gamma < \varphi \langle \mathbf{f}''_j \rangle, \quad 1 \leq j \leq m$$

Now, for  $i \neq j$  we find

$$\begin{aligned} 2\varphi \langle \mathbf{f}'_i, \mathbf{f}'_j \rangle &= 2\varphi \langle \mathbf{f}_i - \mathbf{f}''_i, \mathbf{f}_j - \mathbf{f}''_j \rangle \geq \min \{2\varphi \langle \mathbf{f}_i, \mathbf{f}''_j \rangle, 2\varphi \langle \mathbf{f}''_i, \mathbf{f}_j \rangle, 2\varphi \langle \mathbf{f}''_i, \mathbf{f}''_j \rangle\} \\ &\geq \min \{\varphi \langle \mathbf{f}_i \rangle + \varphi \langle \mathbf{f}''_j \rangle, \varphi \langle \mathbf{f}''_i \rangle + \varphi \langle \mathbf{f}_j \rangle, \varphi \langle \mathbf{f}''_i \rangle + \varphi \langle \mathbf{f}''_j \rangle\} > 2\gamma \end{aligned}$$

so that

$$(11) \quad \varphi\langle f'_i, f'_j \rangle > \gamma, \quad 1 \leq i \neq j \leq m$$

Thus  $f'_1, \dots, f'_m$  are an almost orthogonal system in the  $v(t)$ -dimensional space  $k(e_i)_{i \in J}$ . Assume by way of contradiction that the  $f'_j$  were linearly dependent,  $\sum_1^m \mu_i f'_i = 0$  and not all  $\mu_i = 0$ . Thus, for each

$$r \in \{1, \dots, m\}, 0 = \sum \mu_i \langle f'_i, f'_r \rangle$$

and so for each  $r$

$$\varphi\langle f'_r \rangle + \varphi(\mu_r) = \varphi\left(-\sum_{j \neq r} \mu_j \langle f'_j, f'_r \rangle\right) \geq \min_{j \neq r} \{\varphi(\mu_j) + \varphi\langle f'_j, f'_r \rangle\}.$$

Therefore, by (10) and (11),  $\varphi(\mu_r) > \min_{j \neq r} \{\varphi(\mu_j)\}$  which tells that there is no smallest  $\varphi(\mu_r)$  at all, a contradiction. Therefore,  $f'_1, \dots, f'_m$  are linearly independent and so  $m \leq v(t)$ , QED. By Lemma 27 we thus obtain

**COROLLARY 26.** *The function  $v$  that counts types on an orthogonal basis of  $\mathfrak{E}$  is the same on all bases.*

**VI.3. THE TYPE CONDITION.** Let  $\mathfrak{E}$  be the completion of a  $N_0$ -dimensional space  $\mathfrak{F}$  which is spanned by an orthogonal basis  $(e_i)_{i \in \mathbb{N}}$  that satisfies the type condition (Def. 21).

**LEMMA 27.** *Let  $\mathcal{B} = (u_i)_{i \in \mathbb{N}}$  be a maximal orthogonal family in  $\mathfrak{E}$ . Then  $\mathcal{B}$  satisfies the type condition and  $x = \sum_N \langle x, u_i \rangle \langle u_i \rangle^{-1} u_i$  for all  $x \in \mathfrak{E}$ . In particular, the span of  $\mathcal{B}$  is dense in  $\mathfrak{E}$ .*

*Proof.* The assertion on the type condition follows directly from Lemma 25. Let then  $x \in \mathfrak{E}$ .

$$\begin{aligned} \varphi\langle \langle x, u_i \rangle \langle u_i \rangle^{-1} u_i \rangle &= 2\varphi\langle x, u_i \rangle - \varphi\langle u_i \rangle \geq \varphi\langle x \rangle \\ &\quad + \varphi\langle u_i \rangle - \varphi\langle u_i \rangle = \varphi\langle x \rangle. \end{aligned}$$

Thus the family of vectors  $\langle x, u_i \rangle \langle u_i \rangle^{-1} u_i$  is bounded; in fact, it is a null sequence as  $\mathcal{B}$  satisfies the type condition, hence it is summable as  $\mathfrak{E}$  is complete. Put  $\eta := \sum_N \langle x, u_i \rangle \langle u_i \rangle^{-1} u_i$ . We have  $\langle u_i, \eta - x \rangle = \langle u_i, \eta \rangle - \langle u_i, x \rangle = 0$ , so  $x - \eta = 0$  as  $\mathcal{B}$  is a maximal orthogonal family.  $\square$