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IV.3. THE CLASS  $\mathcal{D}$  OF DEFINITE SPACES. Positive definite forms over ordered fields satisfy the triangle inequality as well as the Cauchy-Schwarz inequality. We therefore set down

*Definition 15.* A definite space is a nondegenerate hermitean space  $(\mathfrak{E}; \langle \cdot, \cdot \rangle)$  over an involutorial division ring  $(k, *)$ ,  $\text{char } k \neq 2$ , that is equipped with a  $*$ -valuation  $\varphi$  that has  $\varphi(2) = 0$  (cf. Remark 35) and that satisfies one (and hence all) of the four statements in Lemma 14. A definite space  $\mathfrak{E}$  will always be considered as a topological vector space, the topology being given by the zero-neighbourhood basis  $\mathcal{U}_\gamma := \{\eta \in \mathfrak{E} \mid \varphi\langle \eta \rangle \geq \gamma\}$ ,  $\gamma \in \Gamma$ . If  $(e_\iota)_{\iota \in I}$  is any family over vectors in  $\mathfrak{E}$  such that the net of all finite ("partial") sums  $\sum e_\iota$  has a limit  $x$  in  $\mathfrak{E}$  then we write  $x = \sum_{\iota \in I} e_\iota$  and call  $(e_\iota)_{\iota \in I}$  summable.

LEMMA 16. Let  $(e_\iota)_{\iota \in I}$  be an orthogonal family in the definite space  $(\mathfrak{E}; \langle \cdot, \cdot \rangle)$  and  $\mathfrak{F}$  its span. For each  $x$  in the topological closure of  $\mathfrak{F}$  we have  $x = \sum_{\iota \in I} \langle x, e_\iota \rangle \langle e_\iota \rangle^{-1} e_\iota$ .

*Proof.* Let  $\mathcal{P}$  be the set of all finite subsets of  $I$ . For  $V \in \mathcal{P}$  we set  $x_V := \sum_{\iota \in V} \langle x, e_\iota \rangle \langle e_\iota \rangle^{-1} e_\iota$ . We have to prove that for each  $\gamma \in \Gamma$  there is  $U \in \mathcal{P}$  such that  $\varphi\langle x - x_V \rangle \geq \varepsilon$  for all  $V$  with  $U \subset V \in \mathcal{P}$ . Now there is  $\eta \in \mathfrak{F}$  with  $\varphi\langle x - \eta \rangle \geq \varepsilon$ . Pick  $U \in \mathcal{P}$  with  $\eta \in \text{span}\{e_\iota \mid \iota \in U\}$ . If  $U \subset V \in \mathcal{P}$  then  $x - x_V \perp x_V - \eta$ , so by "Pythagoras" (Lemma 14 (ii)) we obtain  $\varepsilon \leq \varphi\langle x - \eta \rangle = \min\{\varphi\langle x - x_V \rangle, \varphi\langle x_V - \eta \rangle\} \leq \varphi\langle x - x_V \rangle$ .  $\square$

## V. NECESSARY CONDITIONS IN $\mathcal{D}$ FOR $L_c = L_{\perp\perp}$

The principal result of this section is

THEOREM 17 ([20]). Let  $\mathfrak{E}$  be an infinite dimensional definite space carrying an admissible topology i.e., the topology mentioned in Definition 15 is admissible in the sense of Definition 1; let furthermore  $(e_\iota)_{\iota \in I}$  be an orthogonal family in  $\mathfrak{E}$  such that  $(\varphi\langle e_\iota \rangle)_{\iota \in I}$  has a lower bound in  $\Gamma$ . Then  $\sum_{\iota \in I} e_\iota$  exists.

*Proof.* Let  $\mathfrak{F} := \text{span}\{\langle e_\iota \rangle^{-1} e_\iota - \langle e_0 \rangle^{-1} e_0 \mid \iota \in I\}$ . We first wish to show that  $\langle e_0 \rangle^{-1} e_0$  is not an element of the topological closure  $\overline{\mathfrak{F}}$ . Indeed,

if  $\gamma$  is a lower bound of  $(\varphi\langle e_i \rangle)_{i \in I}$  and if we let  $x := \sum_{i \in U} \lambda_i \langle e_i \rangle^{-1} e_i - \langle e_0 \rangle^{-1} e_0$  be a typical vector of  $\mathfrak{F}$  ( $U$  some finite nonvoid subset of  $I \setminus \{0\}$ ) then we get the inequalities

$$\begin{aligned} \varphi\langle x - \langle e_0 \rangle^{-1} e_0 \rangle &= \varphi\langle (-1 - \sum_U \lambda_i) \langle e_0 \rangle^{-1} e_0 + \sum_U \lambda_i \langle e_i \rangle^{-1} e_i \rangle \\ &= \min_{i \in U} \{2\varphi(-1 - \sum_U \lambda_i) - \varphi\langle e_0 \rangle, 2\varphi(\lambda_i) - \varphi\langle e_i \rangle\} \\ &\leq 2 \min_{i \in U} \{\varphi(-1 - \sum_U \lambda_i), \varphi(\lambda_i)\} - \gamma \leq \varphi(-1) - \gamma = -\gamma. \end{aligned}$$

Thus  $\overline{\mathfrak{F}} \neq \mathfrak{E}$ .

Since  $\mathfrak{F}^{\perp\perp} = \overline{\mathfrak{F}}$  we have  $\mathfrak{F}^{\perp} \neq (0)$ . Pick a non-zero  $x \in \mathfrak{F}^{\perp}$ ; so  $\langle x, e_i \rangle \langle e_i \rangle^{-1} = \langle x, e_0 \rangle \langle e_0 \rangle^{-1}$ . If we assume that  $(e_i)_{i \in I}$  is a maximal orthogonal family then by  $L_c = L_s$  and Lemma 16  $x = \sum_I \langle x, e_i \rangle \langle e_i \rangle^{-1} e_i = \langle x, e_0 \rangle \langle e_0 \rangle^{-1} \sum_I e_i$  and thus  $\sum_I e_i \in \mathfrak{F}^{\perp}$ . If  $(e_i)_{i \in I}$  is not maximal then we write it as a difference of two maximal bounded families: Complete the given family to a maximal orthogonal bounded family  $(e_i)_{i \in J}$ ,  $J \supset I$ , by Zorn's Lemma. For  $i \in J$  let  $\alpha_i := 1 \in k$  when  $i \in I$  and  $\alpha_i := 2$  when  $i \in J \setminus I$ . The two families  $(2e_i)_{i \in J}$ ,  $(\alpha_i e_i)_{i \in J}$  are bounded maximal families to which the previous result may be applied. We get  $\sum_{i \in I} e_i = \sum_{i \in I} (2e_i) - \sum_{i \in I} \alpha_i e_i \in \mathfrak{E}$ .  $\square$

COROLLARY 18. If  $\mathfrak{E}$  and  $(e_i)_{i \in I}$  are as in Theorem 17 then  $(e_i)_{i \in I}$  converges to  $0 \in \mathfrak{E}$ .  $\square$

COROLLARY 19. If  $\mathfrak{E}$  is as in Theorem 17 then the cofinality type of  $\Gamma$  is  $\omega_0$ . In particular, the topology on  $\mathfrak{E}$  satisfies the first countability axiom.  $\square$

COROLLARY 20. If  $\mathfrak{E}$  is as in Theorem 17 then all orthogonal families of non-zero vectors are countable.

*Proof.* Let  $(e_i)_{i \in I}$  be such a family; by multiplying  $e_i$  by a suitable scalar, if necessary, we may assume  $(\varphi\langle e_i \rangle)_{i \in I}$  to be bounded below. Since  $\sum_{i \in I} e_i$  exists by Theorem 17, the sets  $I_\gamma = \{i \in I \mid \varphi\langle e_i \rangle \leq \gamma\}$  are finite for all  $\gamma \in \Gamma$ . Let  $(\gamma_i)_{i \in \mathbb{N}}$  be confinal in  $\Gamma$ . Then  $I = \cup \{I_{\gamma_i} \mid i \in \mathbb{N}\}$  is countable.  $\square$

*Definition 21.* The elements of the group  $\Gamma/2\Gamma$  are called *types*. Let  $T: \Gamma \rightarrow \Gamma/2\Gamma$  be the canonical projection.  $T \circ \varphi$  is constant on the square classes of  $k$  (elements of  $k/k^2$ ) and  $T \circ \varphi \circ \langle \rangle$  is constant on the "punctured"

straight lines in  $E$ . A family  $(e_l)_{l \in I}$  of vectors in  $\mathfrak{E}$  is said to satisfy the *type-condition* iff for all  $(\alpha_l)_{l \in I} \in k^I$  the following holds: if  $(\varphi\langle\alpha_l e_l\rangle)_{l \in I}$  is bounded (below) then  $(\alpha_l e_l)_{l \in I}$  converges to  $0 \in E$ .

**COROLLARY 22.** *Let  $\mathfrak{E}$  be as in Theorem 17.  $\Gamma/2\Gamma$  is infinite. Each orthogonal family in  $\mathfrak{E}$  satisfies the type-condition, equivalently,  $\Gamma/2\Gamma$  satisfies (8) below.  $\square$*

**COROLLARY 23.** *Let  $\mathfrak{E}$  be as in Theorem 17. Then  $k$  is complete.*

*Proof.* By Corollary 19 it suffices to show that a sequence  $(\alpha_i)_{i \in \mathbb{N}}$  with limit  $0 \in k$  is summable. Let  $(e_i)_{i \in \mathbb{N}}$  be maximal orthogonal in  $\mathfrak{E}$  with  $(\varphi\langle e_i \rangle)_{i \in \mathbb{N}}$  bounded below. If  $(\lambda_i)_{i \in \mathbb{N}} \in k^{\mathbb{N}}$  has  $(\varphi(\lambda_i))_{i \in \mathbb{N}}$  bounded below then  $(\lambda_i e_i)_{i \in \mathbb{N}}$  is summable and by continuity of  $\langle \cdot, \cdot \rangle$  we obtain

$$\langle \sum_{\mathbb{N}} \lambda_i e_i, \sum_{\mathbb{N}} e_i \rangle = \sum_{\mathbb{N}} \lambda_i \langle e_i \rangle.$$

Thus, all families  $(\lambda_i \langle e_i \rangle)_{i \in \mathbb{N}}$  with bounded  $(\lambda_i)_{i \in \mathbb{N}}$  are summable.

Pick a strictly monotonic sequence  $(n_i)_{i \in \mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$  with  $u_0 = 0$  and for all  $i \in \mathbb{N}^+$  and all  $m \geq n_i$ :  $\varphi(\alpha_m) > \varphi\langle e_i \rangle$ , and set  $A_i := \sum \{\alpha_j \mid n_i \leq j < n_{i+1}\}$ . The family  $(A_i)_{i \in \mathbb{N}}$  is summable if and only if  $(\alpha_i)_{i \in \mathbb{N}}$  summable and, if the sums exist, these must be equal. If we set  $\lambda_i := A_i \langle e_i \rangle^{-1}$  then, by what we have shown, the family of the  $A_i = \lambda_i \langle e_i \rangle$  is summable.  $\square$

**COROLLARY 24.** *Let  $\mathfrak{E}$  be as in Theorem 17. Then  $\mathfrak{E}$  is complete.*

*Proof.* Let  $(x_i)_{i \in \mathbb{N}}$  be a Cauchy sequence (Corollary 19). For each fixed  $\eta \in \mathfrak{E}$  the map  $x \mapsto \langle \eta, x \rangle$  is uniformly continuous. Hence by Cor. 23 the map  $f: \eta \mapsto \lim_i \langle \eta, x_i \rangle$  is well-defined. As it is a continuous linear map, its kernel is a closed hyper-plane and so  $(L_c(\mathfrak{E}) = L_{\perp\perp}(\mathfrak{E}))$  there is  $a \in \mathfrak{E}$  such that  $f(\eta) = \langle x, a \rangle$ . Let  $N \subseteq \mathbb{N}$  be infinite. Because  $\lim \varphi\langle \eta, a - x_i \rangle = \infty$  for all  $\eta \in \mathfrak{E}$  it follows by systematic use of the Cauchy-Schwarz inequality that  $\{\varphi\langle a - x_i \rangle \mid i \in N\}$  is not bounded above by any  $\gamma \in \Gamma$ . Therefore  $(x_i)_{i \in \mathbb{N}}$  converges to  $a$ .

## VI. SUFFICIENT CONDITIONS IN $\mathscr{D}$ FOR $L_c = L_{\perp\perp}$

**VI.1. ASSUMPTIONS.** In this chapter  $(\mathfrak{E}; \langle \cdot, \cdot \rangle)$  is a definite space in the sense of Definition 15. Of the base field  $k$  we shall furthermore assume