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IV. THE FUNDAMENTAL INEQUALITIES IN DEFINITE SPACES

IV.1. *-VALUATIONS (cf. [14]). Let $(k, *)$ be an involutorial division ring and Γ a totally ordered (additively written) abelian group. A surjective map

$$(7) \quad \varphi: k \rightarrow \Gamma \cup \{\infty\} \quad (a + \infty = \infty \quad \text{for all } a \in \Gamma \cup \{\infty\})$$

is called *-valuation iff (i) $\varphi(x + y) \geq \min\{\varphi(x), \varphi(y)\}$, (ii) $\varphi(xy) = \varphi(x) + \varphi(y)$, (iii) $\varphi(x) = \infty \Leftrightarrow x = 0$, (iv) $\varphi(x) = \varphi(x^*)$.

The set of all $U_\varepsilon := \{x \in k \mid \varphi(x) \geq \varepsilon\}$, $\varepsilon \in \Gamma$, is a neighbourhood basis for a division ring topology on k . In general we think of $(k, *)$ as equipped with this topology.

IV.2. THE INEQUALITIES. Assume that $\text{char } k \neq 2$ and that the valuation in (7) has $\varphi(2) = 0$ (cf. Remark 35). Let $\langle \cdot, \cdot \rangle$ be a hermitean form on a k -space \mathfrak{E} . Assume \mathfrak{E} non-degenerate ($\mathfrak{E}^\perp = (0)$). Recall that we write " $\langle \mathfrak{x} \rangle$ " for $\langle \mathfrak{x}, \mathfrak{x} \rangle$, $\mathfrak{x} \in \mathfrak{E}$. It is useful to know a proof for the following fact

LEMMA 14 ([20]). *The following four statements are equivalent*

- (i) $\forall \mathfrak{x}, \eta \in \mathfrak{E}: \varphi\langle \mathfrak{x} + \eta \rangle \geq \min\{\varphi\langle \mathfrak{x} \rangle, \varphi\langle \eta \rangle\}$ (*triangle inequality*)
- (ii) $\forall \mathfrak{x}, \eta \in \mathfrak{E}: \langle \mathfrak{x}, \eta \rangle = 0 \Rightarrow \varphi\langle \mathfrak{x} + \eta \rangle = \min\{\varphi\langle \mathfrak{x} \rangle, \varphi\langle \eta \rangle\}$
(*"Pythagoras"*)
- (iii) $\forall \mathfrak{x}, \eta \in \mathfrak{E}: \varphi\langle \mathfrak{x}, \eta \rangle \geq \min\{\varphi\langle \mathfrak{x} \rangle, \varphi\langle \eta \rangle\}$ (*"weak Cauchy-Schwarz"*)
- (iv) $\forall \mathfrak{x}, \eta \in \mathfrak{E}: 2\varphi\langle \mathfrak{x}, \eta \rangle \geq \varphi\langle \mathfrak{x} \rangle + \varphi\langle \eta \rangle$ (*"Cauchy-Schwarz"*)

(Notice that each statement implies anisotropy of \mathfrak{E}).

Proof. (i) \Rightarrow (ii): Let $\mathfrak{x} \perp \eta$ and

$$\begin{aligned} \varphi\langle \mathfrak{x} \rangle &\leq \varphi\langle \eta \rangle; \quad \varphi\langle \mathfrak{x} \rangle = \varphi\langle 2\mathfrak{x} \rangle = \varphi\langle (\mathfrak{x} + \eta) \\ &+ (\mathfrak{x} - \eta) \rangle \geq \min\{\varphi\langle \mathfrak{x} + \eta \rangle, \varphi\langle \mathfrak{x} - \eta \rangle\} = \varphi\langle \mathfrak{x} + \eta \rangle \geq \varphi\langle \mathfrak{x} \rangle. \end{aligned}$$

(ii) \Rightarrow (iv): Assume $\mathfrak{x} \neq 0 \neq \eta$. Pick \mathfrak{b} in the span of \mathfrak{x}, η such that

$$\begin{aligned} \mathfrak{x} &= \mathfrak{b} + \lambda\eta, \quad \mathfrak{b} \perp \eta; \quad 2\varphi\langle \mathfrak{x}, \eta \rangle = 2\varphi\langle \mathfrak{b} + \lambda\eta, \eta \rangle = 2\varphi\langle \lambda\eta, \eta \rangle \\ &= 2\varphi\langle \lambda \rangle + 2\varphi\langle \eta \rangle = \varphi\langle \lambda\eta \rangle + \varphi\langle \eta \rangle \geq \varphi\langle \mathfrak{x} \rangle + \varphi\langle \eta \rangle. \end{aligned}$$

(iv) \Rightarrow (iii): trivial

(iii) \Rightarrow (i): straight forward. □

IV.3. THE CLASS \mathcal{D} OF DEFINITE SPACES. Positive definite forms over ordered fields satisfy the triangle inequality as well as the Cauchy-Schwarz inequality. We therefore set down

Definition 15. A definite space is a nondegenerate hermitean space $(\mathfrak{E}; \langle \cdot, \cdot \rangle)$ over an involutorial division ring $(k, *)$, $\text{char } k \neq 2$, that is equipped with a $*$ -valuation φ that has $\varphi(2) = 0$ (cf. Remark 35) and that satisfies one (and hence all) of the four statements in Lemma 14. A definite space \mathfrak{E} will always be considered as a topological vector space, the topology being given by the zero-neighbourhood basis $\mathcal{U}_\gamma := \{\eta \in \mathfrak{E} \mid \varphi\langle \eta \rangle \geq \gamma\}$, $\gamma \in \Gamma$. If $(e_\iota)_{\iota \in I}$ is any family over vectors in \mathfrak{E} such that the net of all finite ("partial") sums $\sum e_\iota$ has a limit x in \mathfrak{E} then we write $x = \sum_{\iota \in I} e_\iota$ and call $(e_\iota)_{\iota \in I}$ summable.

LEMMA 16. Let $(e_\iota)_{\iota \in I}$ be an orthogonal family in the definite space $(\mathfrak{E}; \langle \cdot, \cdot \rangle)$ and \mathfrak{F} its span. For each x in the topological closure of \mathfrak{F} we have $x = \sum_{\iota \in I} \langle x, e_\iota \rangle \langle e_\iota \rangle^{-1} e_\iota$.

Proof. Let \mathcal{P} be the set of all finite subsets of I . For $V \in \mathcal{P}$ we set $x_V := \sum_{\iota \in V} \langle x, e_\iota \rangle \langle e_\iota \rangle^{-1} e_\iota$. We have to prove that for each $\gamma \in \Gamma$ there is $U \in \mathcal{P}$ such that $\varphi\langle x - x_V \rangle \geq \varepsilon$ for all V with $U \subset V \in \mathcal{P}$. Now there is $\eta \in \mathfrak{F}$ with $\varphi\langle x - \eta \rangle \geq \varepsilon$. Pick $U \in \mathcal{P}$ with $\eta \in \text{span}\{e_\iota \mid \iota \in U\}$. If $U \subset V \in \mathcal{P}$ then $x - x_V \perp x_V - \eta$, so by "Pythagoras" (Lemma 14 (ii)) we obtain $\varepsilon \leq \varphi\langle x - \eta \rangle = \min\{\varphi\langle x - x_V \rangle, \varphi\langle x_V - \eta \rangle\} \leq \varphi\langle x - x_V \rangle$. \square

V. NECESSARY CONDITIONS IN \mathcal{D} FOR $L_c = L_{\perp\perp}$

The principal result of this section is

THEOREM 17 ([20]). Let \mathfrak{E} be an infinite dimensional definite space carrying an admissible topology i.e., the topology mentioned in Definition 15 is admissible in the sense of Definition 1; let furthermore $(e_\iota)_{\iota \in I}$ be an orthogonal family in \mathfrak{E} such that $(\varphi\langle e_\iota \rangle)_{\iota \in I}$ has a lower bound in Γ . Then $\sum_{\iota \in I} e_\iota$ exists.

Proof. Let $\mathfrak{F} := \text{span}\{\langle e_\iota \rangle^{-1} e_\iota - \langle e_0 \rangle^{-1} e_0 \mid \iota \in I\}$. We first wish to show that $\langle e_0 \rangle^{-1} e_0$ is not an element of the topological closure $\overline{\mathfrak{F}}$. Indeed,