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## IV. THE FUNDAMENTAL INEQUALITIES IN DEFINITE SPACES

IV.1. \*-VALUATIONS (cf. [14]). Let  $(k, *)$  be an involutorial division ring and  $\Gamma$  a totally ordered (additively written) abelian group. A surjective map

$$(7) \quad \varphi: k \rightarrow \Gamma \cup \{\infty\} \quad (a + \infty = \infty \quad \text{for all } a \in \Gamma \cup \{\infty\})$$

is called \*-valuation iff (i)  $\varphi(x+y) \geq \min\{\varphi(x), \varphi(y)\}$ , (ii)  $\varphi(xy) = \varphi(x) + \varphi(y)$ , (iii)  $\varphi(x) = \infty \Leftrightarrow x = 0$ , (iv)  $\varphi(x) = \varphi(x^*)$ .

The set of all  $U_\varepsilon := \{x \in k \mid \varphi(x) \geq \varepsilon\}$ ,  $\varepsilon \in \Gamma$ , is a neighbourhood basis for a division ring topology on  $k$ . In general we think of  $(k, *)$  as equipped with this topology.

IV.2. THE INEQUALITIES. Assume that  $\text{char } k \neq 2$  and that the valuation in (7) has  $\varphi(2) = 0$  (cf. Remark 35). Let  $\langle \cdot, \cdot \rangle$  be a hermitean form on a  $k$ -space  $\mathfrak{E}$ . Assume  $\mathfrak{E}$  non-degenerate ( $\mathfrak{E}^\perp = (0)$ ). Recall that we write " $\langle \mathfrak{x} \rangle$ " for  $\langle \mathfrak{x}, \mathfrak{x} \rangle$ ,  $\mathfrak{x} \in \mathfrak{E}$ . It is useful to know a proof for the following fact

LEMMA 14 ([20]). *The following four statements are equivalent*

- (i)  $\forall \mathfrak{x}, \mathfrak{y} \in \mathfrak{E}: \varphi\langle \mathfrak{x} + \mathfrak{y} \rangle \geq \min\{\varphi\langle \mathfrak{x} \rangle, \varphi\langle \mathfrak{y} \rangle\}$  (*triangle inequality*)
- (ii)  $\forall \mathfrak{x}, \mathfrak{y} \in \mathfrak{E}: \langle \mathfrak{x}, \mathfrak{y} \rangle = 0 \Rightarrow \varphi\langle \mathfrak{x} + \mathfrak{y} \rangle = \min\{\varphi\langle \mathfrak{x} \rangle, \varphi\langle \mathfrak{y} \rangle\}$  ("Pythagoras")
- (iii)  $\forall \mathfrak{x}, \mathfrak{y} \in \mathfrak{E}: \varphi\langle \mathfrak{x}, \mathfrak{y} \rangle \geq \min\{\varphi\langle \mathfrak{x} \rangle, \varphi\langle \mathfrak{y} \rangle\}$  ("weak Cauchy-Schwarz")
- (iv)  $\forall \mathfrak{x}, \mathfrak{y} \in \mathfrak{E}: 2\varphi\langle \mathfrak{x}, \mathfrak{y} \rangle \geq \varphi\langle \mathfrak{x} \rangle + \varphi\langle \mathfrak{y} \rangle$  ("Cauchy-Schwarz")

(Notice that each statement implies anisotropy of  $\mathfrak{E}$ ).

*Proof.* (i)  $\Rightarrow$  (ii): Let  $\mathfrak{x} \perp \mathfrak{y}$  and

$$\begin{aligned} \varphi\langle \mathfrak{x} \rangle &\leq \varphi\langle \mathfrak{y} \rangle; \quad \varphi\langle \mathfrak{x} \rangle = \varphi\langle 2\mathfrak{x} \rangle = \varphi\langle (\mathfrak{x} + \mathfrak{y}) \\ &+ (\mathfrak{x} - \mathfrak{y}) \rangle \geq \min\{\varphi\langle \mathfrak{x} + \mathfrak{y} \rangle, \varphi\langle \mathfrak{x} - \mathfrak{y} \rangle\} = \varphi\langle \mathfrak{x} + \mathfrak{y} \rangle \geq \varphi\langle \mathfrak{x} \rangle. \end{aligned}$$

(ii)  $\Rightarrow$  (iv): Assume  $\mathfrak{x} \neq 0 \neq \mathfrak{y}$ . Pick  $\mathfrak{b}$  in the span of  $\mathfrak{x}, \mathfrak{y}$  such that

$$\begin{aligned} \mathfrak{x} &= \mathfrak{b} + \lambda\mathfrak{y}, \quad \mathfrak{b} \perp \mathfrak{y}; \quad 2\varphi\langle \mathfrak{x}, \mathfrak{y} \rangle = 2\varphi\langle \mathfrak{b} + \lambda\mathfrak{y}, \mathfrak{y} \rangle = 2\varphi\langle \lambda\mathfrak{y}, \mathfrak{y} \rangle \\ &= 2\varphi(\lambda) + 2\varphi\langle \mathfrak{y} \rangle = \varphi\langle \lambda\mathfrak{y} \rangle + \varphi\langle \mathfrak{y} \rangle \geq \varphi\langle \mathfrak{x} \rangle + \varphi\langle \mathfrak{y} \rangle. \end{aligned}$$

(iv)  $\Rightarrow$  (iii): trivial

(iii)  $\Rightarrow$  (i): straight forward. □

IV.3. THE CLASS  $\mathcal{D}$  OF DEFINITE SPACES. Positive definite forms over ordered fields satisfy the triangle inequality as well as the Cauchy-Schwarz inequality. We therefore set down

*Definition 15.* A definite space is a nondegenerate hermitean space  $(\mathfrak{E}; \langle \cdot, \cdot \rangle)$  over an involutorial division ring  $(k, *)$ ,  $\text{char } k \neq 2$ , that is equipped with a  $*$ -valuation  $\varphi$  that has  $\varphi(2) = 0$  (cf. Remark 35) and that satisfies one (and hence all) of the four statements in Lemma 14. A definite space  $\mathfrak{E}$  will always be considered as a topological vector space, the topology being given by the zero-neighbourhood basis  $\mathfrak{U}_\gamma := \{\eta \in \mathfrak{E} \mid \varphi(\eta) \geq \gamma\}$ ,  $\gamma \in \Gamma$ . If  $(e_i)_{i \in I}$  is any family over vectors in  $\mathfrak{E}$  such that the net of all finite (“partial”) sums  $\sum e_i$  has a limit  $x$  in  $\mathfrak{E}$  then we write  $x = \sum_{i \in I} e_i$  and call  $(e_i)_{i \in I}$  summable.

**LEMMA 16.** Let  $(e_i)_{i \in I}$  be an orthogonal family in the definite space  $(\mathfrak{E}; \langle \cdot, \cdot \rangle)$  and  $\mathfrak{F}$  its span. For each  $x$  in the topological closure of  $\mathfrak{F}$  we have  $x = \sum_{i \in I} \langle x, e_i \rangle \langle e_i \rangle^{-1} e_i$ .

*Proof.* Let  $\mathcal{P}$  be the set of all finite subsets of  $I$ . For  $V \in \mathcal{P}$  we set  $x_V := \sum_{i \in V} \langle x, e_i \rangle \langle e_i \rangle^{-1} e_i$ . We have to prove that for each  $\gamma \in \Gamma$  there is  $U \in \mathcal{P}$  such that  $\varphi(x - x_V) \geq \gamma$  for all  $V$  with  $U \subset V \in \mathcal{P}$ . Now there is  $\eta \in \mathfrak{F}$  with  $\varphi(x - \eta) \geq \gamma$ . Pick  $U \in \mathcal{P}$  with  $\eta \in \text{span}\{e_i \mid i \in U\}$ . If  $U \subset V \in \mathcal{P}$  then  $x - x_V \perp x_V - \eta$ , so by “Pythagoras” (Lemma 14 (ii)) we obtain  $\gamma \leq \varphi(x - \eta) = \min\{\varphi(x - x_V), \varphi(x_V - \eta)\} \leq \varphi(x - x_V)$ .  $\square$

## V. NECESSARY CONDITIONS IN $\mathcal{D}$ FOR $L_c = L_{\perp\perp}$

The principal result of this section is

**THEOREM 17 ([20]).** Let  $\mathfrak{E}$  be an infinite dimensional definite space carrying an admissible topology i.e., the topology mentioned in Definition 15 is admissible in the sense of Definition 1; let furthermore  $(e_i)_{i \in I}$  be an orthogonal family in  $\mathfrak{E}$  such that  $(\varphi(e_i))_{i \in I}$  has a lower bound in  $\Gamma$ . Then  $\sum_{i \in I} e_i$  exists.

*Proof.* Let  $\mathfrak{F} := \text{span}\{\langle e_i \rangle^{-1} e_i - \langle e_0 \rangle^{-1} e_0 \mid i \in I\}$ . We first wish to show that  $\langle e_0 \rangle^{-1} e_0$  is not an element of the topological closure  $\overline{\mathfrak{F}}$ . Indeed,