Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 31 (1985)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: ON A CLASS OF ORTHOMODULAR QUADRATIC SPACES

Autor: Gross, Herbert / Künzi, Urs-Martin

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DOI: https://doi.org/10.5169/seals-54565

be proved *not* to contain orthomodular specimen; we refer to [9]. Here we mention but one result ([9, p. 20]); it has been crucial on the road to Keller's discovery. The idea of its proof is used again in the proof of Theorem 17 below.

THEOREM 13 (Gross-Keller). *Let* k *be a non archimedean ordered field and equipped with its order topology; let* $\langle\, ,\, \rangle$ *be a definite symmetric form on the k-vector space* $\mathfrak{E}$. *Equip* $\mathfrak{E}$ *with the norm topology*

$$(\| \mathfrak{x} \| := \langle \mathfrak{x}, \mathfrak{x} \rangle^{\frac{1}{2}} \in k^{\frac{1}{2}}) .$$

Assume that $\mathfrak{E}$ *contains at least one orthogonal family* $(\mathfrak{e}_i)_{i \in \mathbf{N}}$ *that is bounded, i.e. for suitable* $\alpha, \beta \in k$

$$0 < \alpha \leqslant \langle \mathfrak{e}_i, \mathfrak{e}_i \rangle \leqslant \beta \quad (i \in \mathbf{N}) \tag{6}$$

Then $L_{\perp\perp}(\mathfrak{E}) \underset{\neq}{\subset} L_c(\mathfrak{E})$.

III. Keller's Example

The authors of [9] lamented about the "irksome" condition (6) which, indeed, need not be satisfied (*loc. cit.*, p. 89). Keller finally noticed that (6) pointed at the very crux of the matter. He considered the transcendental extension $k_0 = \mathbf{Q}(X_i)_{i \in \mathbf{N}}$ with the unique ordering that has $X_0 > q$ for all $q \in \mathbf{Q}$ and $X_i^n < X_{i+1}$ for all i and all n; then he let k be the completion of k_0 by means of Cauchy sequences. $\mathfrak{E}$ is the linear k-space of all $(y_i)_{i \in \mathbf{N}} \in k^{\mathbf{N}}$ such that $\sum_{\mathbf{N}} y_i^2 X_i$ exists (addition and scalar multiplication component wise) and $\langle (y_i)_{i \in \mathbf{N}}, (z_i)_{i \in \mathbf{N}} \rangle := \sum_{\mathbf{N}} y_i z_i X_i$. Original and ingenious arguments given in [18] establish orthomodularity of $\mathfrak{E}$. (This also follows from our Theorem 36 below.)

Gross noticed that Keller's construction works for valued fields ([6, 7, 20]). An example is also contained in [14, p. 237]).

Keller's choice of a field over which one can build orthomodular spaces has been good: as our results show his space exhibits the typical properties of an orthomodular space with an admissible topology (cf. Remark 29 below).