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**Autor:** Gross, Herbert / Künzi, Urs-Martin  
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Orthomodular lattices that derive from orthomodular quadratic spaces make up only a fraction of abstract orthomodular lattices (refer to [13, 16, 17]). The orthomodular law (4) is exceedingly enigmatic even if attention is restricted to orthomodular quadratic spaces. The complexity of the orthomodular conundrum does not surprise us anymore.

## II. RESULTS ON ORTHOMODULAR SPACES PRIOR TO KELLER'S DISCOVERY

II.1. RESULTS WITHOUT TOPOLOGICAL RESTRICTIONS ON  $\mathfrak{E}$ . We begin with a classic ([1]).

**THEOREM 4** (Amemiya-Araki-Piron). *Let  $k$  be one of  $\mathbf{R}, \mathbf{C}, \mathbf{H}$  and  $\mathfrak{E}$  an infinite-dimensional  $k$ -vector space equipped with a positive definite hermitean form  $\langle \cdot, \cdot \rangle$  (relative to the usual involution  $*$  in  $k$ ). Then  $\mathfrak{E}$  is orthomodular iff  $\mathfrak{E}$  is complete as a normed space*

$$(\|x\| := \langle x, x \rangle^{\frac{1}{2}}),$$

*i.e. iff  $\mathfrak{E}$  is a Hilbert space.*

If, in the setting of Thm. 4, we pass to subfields of  $k$  then the same conclusion can be drawn although the proof is much more tricky [9]:

**THEOREM 5** (Gross-Keller). *Let  $k$  be an archimedean (Baer-)ordered  $*$ -field ([14, p. 219]) and  $\mathfrak{E}$  an infinite dimensional  $k$ -vector space equipped with a positive definite hermitean form. Then the following are equivalent*

- (i)  $k$  is one of  $\mathbf{R}, \mathbf{C}, \mathbf{H}$  and  $\mathfrak{E}$  is a Hilbert space
- (ii)  $L_s(\mathfrak{E}) = L_{\perp\perp}(\mathfrak{E})$  i.e.  $\mathfrak{E}$  is orthomodular
- (iii)  $L_c(\mathfrak{E}) = L_{\perp\perp}(\mathfrak{E})$  ( $c$  refers to the norm  $\|x\| := \langle x, x \rangle^{\frac{1}{2}} \in k^{\frac{1}{2}}$ )
- (iv)  $L_s(\mathfrak{E}) = L_{\perp\perp}(\mathfrak{E}) = L_c(\mathfrak{E})$ .

*Remark 6.* In [24] sequence spaces  $\mathfrak{E} := \ell_2(k)$  for  $k \subset \mathbf{H}$  are considered and equipped with hermitean maps (not forms)  $\mathfrak{E} \times \mathfrak{E} \rightarrow \mathbf{H}$ . Again, the lattice of  $\perp$ -closed subspaces in  $\mathfrak{E}$  is orthomodular iff  $k = \mathbf{R}, \mathbf{C}$ , or  $\mathbf{H}$ .

Another attempt to chance upon new orthomodular forms is to replace the reals by the non-archimedean ordered field  $*\mathbf{R}$ , a non-standard model of  $\mathbf{R}$ . However [28]:

**THEOREM 7 (Morash).** *The inner product on  $\mathfrak{H} = \ell_2(\mathbf{R})$  induces a positive definite symmetric bilinear form  ${}^*\mathfrak{H} \times {}^*\mathfrak{H} \rightarrow {}^*\mathbf{R}$ ; here  ${}^*\mathfrak{H}$  is the set (linear  ${}^*\mathbf{R}$ -space) of equivalence classes in  $\mathfrak{H}^{\mathbf{N}}$  induced by the free ultra filter  $U$  on  $\mathbf{N}$  used to define  ${}^*\mathbf{R}$ . The lattice  $L_{\perp\perp}({}^*\mathfrak{H})$  is complete but not orthomodular.*

*Remark 8.* In [28] it is also shown that the ultra filter construction applied to a product of lattices isomorphic to  $L_{\perp\perp}(\ell_2(\mathbf{R}))$  leads to an orthomodular lattice that, alas, is not complete. This loss of completeness, incidentally, is *the* (only) obstacle on the way to an easy (ultrafilter construction + Theorem 3) existence proof for orthomodular spaces different from Hilbert space.

A rather general theorem is ([33]):

**THEOREM 9 (Wilbur).** *Let  $(k, *)$  be commutative and such that for each  $*$ -symmetric element  $\lambda \in k$  there is  $\alpha \in k$  with  $\lambda = \pm \alpha\alpha^*$ . If  $\mathfrak{E}$  is an orthomodular space over  $k$ ,  $\dim \mathfrak{E}$  infinite, then  $k = \mathbf{R}$  or  $\mathbf{C}$  with  $*$  the identity or the usual conjugation, respectively (so  $\mathfrak{E}$  is a Hilbert space).*

*Remark 10.* The formulation of Thm. 9 in [33] also admits skew  $(k, *)$  with one additional assumption. However, by Dieudonné's Lemma ([10 p. 18])  $(k, *)$  must then be a quaternion algebra with  $*$  the usual conjugation.

Wilbur's result is generalized to ordered  $*$ -fields in [14, § 6].

Hermitean spaces that are orthogonal sums of finite dimensional subspaces are called *diagonal*; subspaces of diagonal spaces are termed *prediagonal*. There is a full-fledged theory about prediagonal spaces of infinite dimensions. Deplorably, we have ([9]):

**THEOREM 11 (Gross-Keller).** *Let  $\dim \mathfrak{E} \geq \aleph_0$ . If  $\mathfrak{E}$  is prediagonal then it is not orthomodular. Thus, in particular,  $\dim \mathfrak{E} > \aleph_0$  if  $\mathfrak{E}$  is orthomodular.*

Orthomodularity of a space  $\mathfrak{E}$  has strange consequences for the base field of  $\mathfrak{E}$ . We just mention one of several [9, p. 15].

**THEOREM 12 (Gross-Keller).** *If  $\text{card } k < 2^{\aleph_0}$  then an infinite dimensional  $k$ -space  $\mathfrak{E}$  cannot be orthomodular.*

**II.2. A RESULT ON SPACES  $\mathfrak{E}$  EQUIPPED WITH AN ADMISSIBLE TOPOLOGY.** Certain well known classes of spaces  $\mathfrak{E}$  that carry admissible topologies can

be proved *not* to contain orthomodular specimen; we refer to [9]. Here we mention but one result ([9, p. 20]); it has been crucial on the road to Keller's discovery. The idea of its proof is used again in the proof of Theorem 17 below.

**THEOREM 13 (Gross-Keller).** *Let  $k$  be a non archimedean ordered field and equipped with its order topology; let  $\langle , \rangle$  be a definite symmetric form on the  $k$ -vector space  $\mathfrak{E}$ . Equip  $\mathfrak{E}$  with the norm topology*

$$(\|x\| := \langle x, x \rangle^{\frac{1}{2}} \in k^{\frac{1}{2}}).$$

*Assume that  $\mathfrak{E}$  contains at least one orthogonal family  $(e_i)_{i \in \mathbf{N}}$  that is bounded, i.e. for suitable  $\alpha, \beta \in k$*

$$(6) \quad 0 < \alpha \leq \langle e_i, e_i \rangle \leq \beta \quad (i \in \mathbf{N})$$

*Then  $L_{\perp \perp}(\mathfrak{E}) \subsetneq L_c(\mathfrak{E})$ .*

### III. KELLER'S EXAMPLE

The authors of [9] lamented about the "irksome" condition (6) which, indeed, need not be satisfied (*loc. cit.*, p. 89). Keller finally noticed that (6) pointed at the very crux of the matter. He considered the transcendental extension  $k_0 = \mathbf{Q}(X_i)_{i \in \mathbf{N}}$  with the unique ordering that has  $X_0 > q$  for all  $q \in \mathbf{Q}$  and  $X_i^n < X_{i+1}$  for all  $i$  and all  $n$ ; then he let  $k$  be the completion of  $k_0$  by means of Cauchy sequences.  $\mathfrak{E}$  is the linear  $k$ -space of all  $(y_i)_{i \in \mathbf{N}} \in k^{\mathbf{N}}$  such that  $\sum_{\mathbf{N}} y_i^2 X_i$  exists (addition and scalar multiplication component wise) and  $\langle (y_i)_{i \in \mathbf{N}}, (z_i)_{i \in \mathbf{N}} \rangle := \sum_{\mathbf{N}} y_i z_i X_i$ . Original and ingenious arguments given in [18] establish orthomodularity of  $\mathfrak{E}$ . (This also follows from our Theorem 36 below.)

Gross noticed that Keller's construction works for valued fields ([6, 7, 20]). An example is also contained in [14, p. 237]).

Keller's choice of a field over which one can build orthomodular spaces has been good: as our results show his space exhibits the typical properties of an orthomodular space with an admissible topology (cf. Remark 29 below).