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ON A CLASS OF ORTHOMODULAR QUADRATIC SPACES

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0. INTRODUCTION

The most important property of the classical Hilbert space

$$\mathfrak{H} = \ell_2 = \{(\lambda_i)_{i \in \mathbf{N}} \mid \lambda_i \in \mathbf{R}\,,\ \Sigma\lambda_i^2 < \infty\}$$

is expressed by the projection theorem: the orthogonal complement $\mathfrak{X}^\perp$ of a closed linear subspace $\mathfrak{X}$ is a linear supplement, in formulae

$$(P_1) \qquad \mathfrak{X} = \overline{\mathfrak{X}} \Rightarrow \mathfrak{H} = \mathfrak{X} \oplus \mathfrak{X}^\perp$$

In the space $\mathfrak{H}$ it happens that precisely those linear subspaces $\mathfrak{X}$ are closed which coincide with their bi-orthogonals, $\mathfrak{X} = \overline{\mathfrak{X}} \Leftrightarrow \mathfrak{X} = (\mathfrak{X}^\perp)^\perp$ ("$\mathfrak{X}$ is $\perp$-closed"). Therefore we may express the projection theorem here in the following purely algebraic way

$$(P_2) \qquad \mathfrak{X} = (\mathfrak{X}^\perp)^\perp \Rightarrow \mathfrak{H} = \mathfrak{X} \oplus \mathfrak{X}^\perp$$

If, in the following, $\mathfrak{H}$ is any vector space over a division ring k and equipped with a hermitean form, then $\mathfrak{H}$ is called orthomodular if (P_2) holds for all linear subspaces $\mathfrak{X}$ of $\mathfrak{H}$.

The problem is to determine what orthomodular spaces there are besides classical Hilbert space (over $k = \mathbf{R}$, $\mathbf{C}$, $\mathbf{H}$). Notice that finite dimensional spaces are uninteresting in this connection because then validity of the projection theorem coincides with non-isotropy of the form. The first infinite dimensional orthomodular space different from the classical ones was discovered in 1979 by H. A. Keller [10 p. 3; 18].

We adduce the following motivations for the study of orthomodular spaces.

§ *1.* The requirement (P_2) on a hermitean space is an extraordinarily strong one. For years the endeavour of a number of people was directed towards proving that there are no examples other than classical Hilbert space [1, 9, 14, 25, 27, 33]. Indeed, all of the prominent Hilbert-like quadratic spaces discussed in the literature could be shown not to be orthomodular (See Sections II.1, II.2 below). As we now know a multitude of orthomodular spaces — there are examples for any characteristic of k — the question of what really lurks behind the projection theorem has become very interesting. The problem to determine *all* hermitean spaces with (P_2) is far from being solved. Although no topologies are involved in (P_2), all methods for the construction of orthomodular spaces that are known are based on topological considerations. The problem raises difficult questions concerning fields.

§ *2.* The Clifford algebras of certain orthomodular spaces H over k are ([6, 7]) normed k-algebras that are division rings (*-valued division rings in the sense of [14]). As the form on H has a canonical extension to its Clifford algebra (char $k \neq 2$), we obtain here a rather interesting class of division algebras that are infinite-dimensional over their centers. These division algebras, as hermitean spaces, are not orthomodular but they can be embedded into orthomodular spaces.

§ *3.* Let $\mathfrak{H}$ be Keller's space of [18] and $\mathscr{B}(\mathfrak{H})$ the algebra of bounded operators on $\mathfrak{H}$. There is hope to chance upon interesting rings of operators. Keller has given examples [19] of self-adjoint $A \in \mathscr{B}(\mathfrak{H})$ that share, among others, the following properties. The von Neumann algebra $\{A\}'$ (centralizer) is commutative; it is however — in contrast to the classical case — irreducible, A has no invariant subspaces. In these examples the arithmetic properties of k play a decisive role. One should first settle the problem whether all $\{A\}'$ with $A \in \mathscr{B}(\mathfrak{H})$ self-adjoint turn out commutative.

§ *4.* In the lattice theoretic viewpoint in physics introduced by G. Birkhoff and J. von Neumann ([4]) the experimentally verifiable propositions about a physical system are identified with the elements of an orthocomplemented lattice (Sec. I.2). On this lattice observables and states can be defined. In quantum physics one assumes that this lattice is the lattice $L_{\perp\perp}(\mathfrak{H})$ of an orthomodular space $\mathfrak{H}$ (or products of such lattices if super selection rules are present). If, for example, it could be made plausible that $\mathfrak{H}$ is over an archimedean ordered field and definite then by Theorem 5 $\mathfrak{H}$ would be a classical Hilbert space (as desired). In our opinion, the main use of Keller's discovery, as far as "quantum logic" is concerned, is to let the axiom that the logic be the usual Hilbert space structure appear even more *ad hoc* than is generally admitted. The base field of Keller's space $\mathfrak{H}$ is non-archimedean ordered. The frequently heard observation that scales on measuring devices in the laboratory are by necessity archimedean ordered is besides the point, for, scales are not connected with the division ring underlying the space $\mathfrak{H}$ but with the range $\mathbf{R}$ of the probability distributions

$$f: L_{\perp\perp}(\mathfrak{H}) \to [0, 1] \subset \mathbf{R}$$

that thrive on the lattice $L_{\perp\perp}(\mathfrak{H})$. Remarkably enough, there is a lavish supply of real valued probability distributions on $L_{\perp\perp}(\mathfrak{H})$ for our non-classical orthomodular spaces $\mathfrak{H}$ in spite of the teratological nature of the base fields (cf. Problem 7 in XIII). *Independent of any axiomatics there is the fascinating mathematical problem to classify these probability distributions.* No approach à la Gleason is possible here [8].

The present paper is meant as an introduction to the topic of orthomodular quadratic spaces. Attention is restricted to hermitean spaces $(\mathfrak{E}; \langle\ ,\ \rangle)$ over valued fields or ordered fields. Let $\mathscr{E}$ be the class of all spaces $\mathfrak{E}$ which admit a vector space topology that makes $\langle\ ,\ \rangle$ continuous (Section VIII). For expository purposes our main interest here is in the subclass $\mathscr{D} \subset \mathscr{E}$ of all "definite" spaces (Definition 15): these are the spaces $\mathfrak{E}$ where a norm defined on $\mathfrak{E}$ via the form $\langle\ ,\ \rangle$ and the valuation (ordering respectively) satisfies a Cauchy-Schwarz type inequality (Section IV). In both classes $\mathscr{D}$, $\mathscr{E}$ the spaces satisfying (P_1) can be characterized (Theorems 28, 34, 36); these spaces satisfy (P_2) as well. This characterization allows to construct orthomodular spaces at will.

We further give a survey of some older results related to orthomodular spaces (Section II). We also append a list of open problems.

I. Orthomodular spaces (Terminology)

I.1 Conventions for the whole paper: In this paper we consider left vector spaces $\mathfrak{E}$ over division rings k with involution $\alpha \mapsto \alpha^*$ (antiautomorphism of k whose square is the identity). $\mathfrak{E}$ is equipped with an anisotropic hermitean form $\langle\ ,\ \rangle$; thus by definition for all

$$\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in \mathfrak{E}, \alpha \in k:$$

$$\langle \alpha\mathfrak{a}+\mathfrak{b}, \mathfrak{c}\rangle = \alpha\langle \mathfrak{a}, \mathfrak{c}\rangle + \langle \mathfrak{b}, \mathfrak{c}\rangle, \langle \mathfrak{a}, \mathfrak{b}\rangle = \langle \mathfrak{b}, \mathfrak{a}\rangle^*, \langle \mathfrak{a}, \mathfrak{a}\rangle = 0 \quad \text{iff} \quad \mathfrak{a} = 0.$$

We shall often abreviate "$\langle \mathfrak{a}, \mathfrak{a}\rangle$" by "$\langle \mathfrak{a}\rangle$". If $\mathfrak{E}$ is infinite dimensional there are always subspaces $\mathfrak{F}$ that are properly contained in their biorthogonals $\mathfrak{F}^{\perp\perp} := (\mathfrak{F}^{\perp})^{\perp}$ [10; Lemma 3, p. 20]. Let $L(\mathfrak{E})$ be the set of all linear subspaces of $\mathfrak{E}$ and