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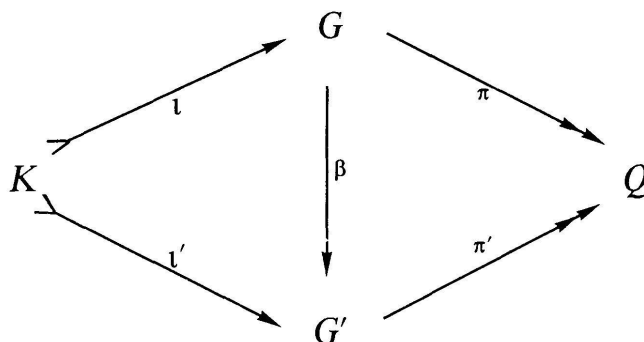
1. AN EXACT SEQUENCE

Fixing two groups K and Q , we consider extensions G with kernel K and quotient Q . (The phraseology is intended to evade the "of K by Q " versus "of Q by K " controversy.) Strictly speaking, K is only isomorphic to the kernel, for we take an extension to be a short exact sequence of groups

$$K \xrightarrow{\iota} G \xrightarrow{\pi} Q,$$

often referring to this simply as " G ".

Two extensions G, G' then are *equivalent* (also known as *congruent*) if there exists a (necessarily bijective) homomorphism $\beta: G \rightarrow G'$ making



commute.

The set of equivalence classes, $\mathcal{E}xt(Q, K)$, is a pointed set in that it admits a distinguished element (basepoint), namely the class of the *trivial extension*

$$K \xrightarrow{in_1} K \times Q \xrightarrow{pr_2} Q.$$

It is usual either to consider more tractable subsets of this set or to specialise to the case of abelian K , so as to obtain richer algebraic structure. However here we look at $\mathcal{E}xt$ in full generality. We determine it to the extent of placing this set in an exact sequence of pointed sets. (Recall that a sequence of pointed set functions

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is exact if $f(A) = g^{-1}(c_0)$, where c_0 is the basepoint of C .) For discussion of naturality of the sequence, we observe that the pointed set functor $\mathcal{E}xt(,)$ is contravariant in the quotient group via the existence of induced

(pulled-back) extensions. On the other hand, in the absence of a commutativity condition it fails to be a (covariant) functor in the kernel. (More on this, later.)

PROPOSITION 1.1. *There is an exact sequence of pointed sets*

$$H^2(Q; Z(K)) \xrightarrow{A} \mathcal{E}xt(Q, K) \xrightarrow{B} \text{Hom}(Q, \text{Out}(K)) \xrightarrow{\Gamma} \coprod_{\alpha} H^3(Q; \{Z(K)\}_{\alpha})$$

where the functions A , B , Γ are defined below.

Proof. First, an explanation of notation. $H^2(Q; Z(K))$ refers to cohomology with trivial coefficients $Z(K)$, the centre of K . On the other hand, $\{\}_{\alpha}$ indicates that the coefficients in $H^3(Q; \{Z(K)\}_{\alpha})$ may be non-trivial, corresponding to a non-trivial homomorphism α from Q to the group $\text{Aut}(Z(K))$ of all automorphisms of $Z(K)$. Cohomology groups, being abelian, have 0 as natural basepoint; \coprod refers to the coproduct in the category of pointed sets, that is, the one-point union obtained by identifying every 0 in the disjoint union. In this case the union is taken over all possible choices of local systems of coefficients; in other words, is indexed by

$$\text{Hom}(Q, \text{Aut}(Z(K))).$$

Finally, $\text{Out}(K)$ denotes the outer automorphism group of K , the quotient of $\text{Aut}(K)$ by its group $\text{Inn}(K)$ of inner automorphisms.

Although this result may be deduced from [9] (see also [15] ch. IV, [11]), I have chosen to outline a more geometric, less ad hoc treatment here. (Equivalence of the corresponding functions occurring in the different approaches has been verified in [13].)

It is of course a standard fact (recaptured below) that $H^2(Q; Z(K))$ corresponds to the subset of $\mathcal{E}xt(Q, Z(K))$ comprising central extensions. (A further topological proof, in the spirit of some of the discussion below, is presented in [2 ch. 8]. That treatment also permits a topological proof of the fact [9] that our function A generalises, to provide a bijection of each inverse image under B with the corresponding $H^2(Q; \{Z(K)\})$.)

The function A is usefully considered in somewhat fuller generality. Therefore let $\tau: Z \rightarrow L$ be a group homomorphism with domain abelian and image central in L . We define $A: H^2(Q; Z) \rightarrow \mathcal{E}xt(Q, L)$ as follows. Given a central extension $Z \xrightarrow{\iota'} E \xrightarrow{\phi'} Q$ representing an equivalence class $[\phi] \in H^2(Q; Z)$, let its image under A be the class of the extension

$$L \xrightarrow{\iota''} L \times E / \tilde{Z} \xrightarrow{\phi''} Q$$

Here the subgroup \tilde{Z} of $L \times E$ consists of all pairs $(\tau(z), \iota'(z^{-1}))$, $z \in Z$, and is normal precisely because $\tau(Z)$ and $\iota'(Z)$ are both central. The homomorphisms ι'' and ϕ are the predictable ones: $\iota''(x) = (x, 1)$ and $\phi''(x, e) = \phi'(e)$. The various checks, for example that ϕ'' , then A , is well-defined, are straightforward and assigned to the reader. Our proof that A is injective follows the definition of B given below. Note (for (1.2) below) that when L is abelian the resulting extension is central, so that A may be regarded as a map

$$H^2(Q; Z) \rightarrow H^2(Q; L) \hookrightarrow \text{Ext}(Q, L).$$

In this form, it reduces to the Baer construction, which coincides with the obvious cohomological homomorphism

$$\tau_*: H^2(Q; Z) \rightarrow H^2(Q; L).$$

The function B is often referred to as the *coupling* [11 p. 65]. For a given extension $K \xrightarrow{\iota} G \xrightarrow{\pi} Q$ it comes from conjugation in K by inverse images in G of elements in Q . Such inverse images being determined only up to multiplication by elements of $\iota(K)$, the G -conjugation automorphism of K is defined only modulo $\text{Inn}(K)$. Again, it is simple to check that B is an invariant of equivalence and thus well-defined.

Now observe that conjugation by $K \times E/\tilde{Z}$ on $\iota''(K)$ has the same effect as K -conjugation. Therefore $B \circ A$ is trivial. If, on the other hand, $K \xrightarrow{\iota} G \xrightarrow{\pi} Q$ induces trivial $Q \rightarrow \text{Out}(K)$, then G coincides with the kernel $\iota K \cdot C_G(\iota K)$ of the trivial composition of homomorphisms in the commuting square

$$\begin{array}{ccc} G & \rightarrow & \text{Aut}(K) \\ \pi \downarrow & & \downarrow \\ Q & \xrightarrow{B\pi} & \text{Out}(K). \end{array}$$

So

$$Q \cong \iota K \cdot C_G(\iota K) / \iota K \cong C_G(\iota K) / \iota Z(K);$$

in other words, there is a central extension

$$Z(K) \xrightarrow{\iota} C_G(K) \xrightarrow{\pi} Q.$$

From the isomorphism

$$K \times C_G(\iota K)/\tilde{Z} \rightarrow G$$

$$(k, g) \mapsto kg$$

we infer that $A[\pi] = [\pi]$, as required for exactness at $\mathcal{E}xt(Q, K)$. Again, if we begin with a central extension $Z(K) \xrightarrow{\iota'} G \xrightarrow{\phi'} Q$, then the extension $K \xrightarrow{\iota''} K \times E/\tilde{Z} \xrightarrow{\phi''} Q$ representing $A[\phi']$ has $Z(K) \xrightarrow{\iota''} C_{K \times E/\tilde{Z}}(\iota'' K) \xrightarrow{\phi''} Q$ equivalent to ϕ' . Thus A is a bijection onto $\text{Ker } B$, with inverse given by restriction.

We turn now to the definition of the function Γ . At this stage classifying spaces (of topological monoids in the case of the set of self-homotopy equivalences $\mathcal{E}(X)$ and its basepoint-preserving counterpart $\mathcal{E}(X; x_0)$, otherwise of discrete groups) enter the picture. From Corollary A.5 there is a fibration

$$\mathcal{K}(Z(K), 2) \rightarrow B\mathcal{E}(X) \rightarrow B\text{Out}(K)$$

where $X = BK = \mathcal{K}(K, 1)$. A homomorphism $\psi: Q \rightarrow \text{Out}(K)$ induces $B\psi: BQ \rightarrow B\text{Out}(K)$.

$$\begin{array}{ccc} & \mathcal{K}(Z(K), 2) & \\ & \downarrow & \\ & B\mathcal{E}(X) & \\ & \downarrow & \\ BQ & \rightarrow & B\text{Out}(K) \end{array}$$

The question as to when $B\psi$ lifts to a map $BQ \rightarrow B\mathcal{E}(X)$ (making the above triangle commute) is solved by standard obstruction theory (e.g. [23 VI (6.14)]), which asserts that there is an element of $H^3(BQ; \{Z(K)\}) = H^3(Q; \{Z(K)\})$, uniquely determined by ψ and therefore safely labelled as $\Gamma\psi$, whose vanishing is equivalent to the existence of the desired lifting. (Note that the local coefficient system $\{Z(K)\}$ is also determined by ψ via its composition with the restriction homomorphism $\text{Out}(K) \rightarrow \text{Aut}(Z(K))$.) Now our present claim is that $\Gamma\psi$ vanishes precisely when ψ is derived, via B , from a group extension. The link between these assertions is provided by the universality of the fibration

$$BK \rightarrow B\mathcal{E}(X; x_0) \rightarrow B\mathcal{E}(X)$$

(e.g. [7]). That is, every fibration with fibre BK is induced from this one by a map of its base space into $B\mathcal{E}(X)$. So liftings $BQ \rightarrow B\mathcal{E}(X)$ of $B\psi$ correspond one-to-one with fibrations $BK \rightarrow BG \rightarrow BQ$. (The homotopy exact sequence here shows that the total space must be a $\mathcal{K}(G, 1)$.) Finally, since the fundamental group functor is left inverse to the classifying space functor, fibrations of this form correspond one-to-one to extensions $K \twoheadrightarrow G \twoheadrightarrow Q$. This clinches exactness at $\text{Hom}(Q, \text{Out}(K))$.

In fact, the argument shows more, for it reveals that the first three terms of (1.1) are none other than those of the exact sequence

$$1 \rightarrow [BQ, \mathcal{K}(Z(K), 2)] \rightarrow [BQ, B\mathcal{E}(X)] \rightarrow [BQ, \text{BOut}(K)]$$

arising from the fibration (A.5) (where the first term, $[BQ, \Omega\text{BOut}(K)]$, is trivial because $\Omega\text{BOut}(K) = \text{Out}(K)$ is discrete). Although this does not yield that A is injective (merely that its kernel is trivial), it does provide a topological proof that $H^2(Q; Z(K)) = [BQ, \mathcal{K}(Z(K), 2)]$ maps with trivial kernel onto $\text{Ker } B$, which we have seen corresponds to the set of equivalence classes of central extensions with quotient Q and kernel $Z(K)$.

We now take up the matter of naturality of this sequence in the kernel K (naturality in the quotient being regarded as obvious). This has significant ramifications for us later on.

PROPOSITION 1.2. *Let H be a characteristic subgroup of K . Then the quotient homomorphism $\kappa: K \rightarrow K/H$ induces a map of exact sequences*

$$\begin{array}{ccccc} H^2(Q; Z(K)) & \xrightarrow{A} & \mathcal{E}\text{xt}(Q, K) & \xrightarrow{B} & \text{Hom}(Q, \text{Out}(K)) \\ \downarrow \kappa_* & & \downarrow \kappa_* & & \downarrow \kappa_* \\ H^2(Q; Z(K/H)) & \xrightarrow{A} & \mathcal{E}\text{xt}(Q, K/H) & \xrightarrow{B} & \text{Hom}(Q, \text{Out}(K/H)). \end{array}$$

Moreover, if $Z(K) \leq H$ and $H/Z(K)$ is normal when regarded as a subgroup of $\text{Aut}(K)$, then there is a partial splitting

$$\Delta: \text{Hom}(Q, \text{Out}(K)) \rightarrow \mathcal{E}\text{xt}(Q, K/H)$$

such that

$$\Delta \circ B = \kappa_* \quad \text{and} \quad B \circ \Delta = \kappa_*.$$

Note that the condition on $H/Z(K)$ is clearly satisfied whenever $H/Z(K)$ is characteristic in $K/Z(K) = \text{Inn}(K)$, as, for example, happens when H is a member of the upper central series of K .

Proof. The cohomological map κ_* has been discussed above; its existence relies only on the normality of H in K . For the $\mathcal{E}xt$ map, let $[\pi]$ represent an extension $K \twoheadrightarrow G \xrightarrow{\pi} Q$. Then $\kappa_*[\pi]$ is defined to be the equivalence class of the extension $K/H \twoheadrightarrow G/H \rightarrow Q$. Here one needs H characteristic in K in order that H be normal in G . Also, when H is characteristic in K there is a canonical homomorphism $\hat{\kappa}: \text{Out}(K) \rightarrow \text{Out}(K/H)$. So $\kappa_*: \text{Hom}(Q, \text{Out}(K)) \rightarrow \text{Hom}(Q, \text{Out}(K/H))$ is given simply by composition with $\hat{\kappa}$. Verification of commutativity of the two squares is a tedious but uncomplicated exercise.

The map Δ is more interesting. It can be viewed as the composition of two of the three constructions on extensions already presented. Beginning with the standard extension

$$K/Z(K) = \text{Inn}(K) \twoheadrightarrow \text{Aut}(K) \twoheadrightarrow \text{Out}(K),$$

we obtain by assumption a second extension

$$K/H \twoheadrightarrow \text{Aut}(K) / (H/Z(K)) \twoheadrightarrow \text{Out}(K),$$

and then pull it back over a given homomorphism $\psi: Q \rightarrow \text{Out}(K)$ to obtain as $\Delta(\psi)$ the induced extension with quotient Q and kernel K/H . The check of commutativity of the two triangles formed is again routine. (In the case $H=Z(K)$, Rose [20] calls the values of Δ *sited extensions*.)

One is tempted to speculate on the existence of a map κ_* at the H^3 level. However this first requires a map of coefficient systems. There is in general no function $\text{Aut}(Z(K)) \rightarrow \text{Aut}(Z(K/H))$ such that the square

$$\begin{array}{ccc} \text{Out}(K) & \rightarrow & \text{Aut}(Z(K)) \\ \downarrow & & \downarrow \\ \text{Out}(K/H) & \rightarrow & \text{Aut}(Z(K/H)) \end{array}$$

(whose horizontal maps are given by restriction) commutes, as may be seen by reference to the example where K is the centreless alternating group A_4 and H is the four-group, a characteristic subgroup. For then $\text{Aut}(Z(K))$ is trivial, while

$$\text{Out}(K) \rightarrow \text{Out}(K/H) \cong \text{Aut}(Z(K/H))$$

is surjective and non-trivial (see, for example, [22]).

An immediate consequence of (1.2) is familiar (for example [20]).

COROLLARY 1.3. *If $Z(K) = 1$, then B and Δ are inverse bijections.*

In particular, every extension with kernel K is induced from the extension $K \rightarrow \text{Aut}(K) \rightarrow \text{Out}(K)$ by a homomorphism into $\text{Out}(K)$.

Another sense in which B admits a partial inverse is provided by the semi-direct product construction (described in for example [21 Theorem 9.9]). This may be regarded as an injection $E: \text{Hom}(Q, \text{Aut}(K)) \rightarrow \mathcal{E}\text{xt}(Q, K)$, which evidently makes the triangle

$$\begin{array}{ccc} & \text{Hom}(Q, \text{Aut}(K)) & \\ E \swarrow & \downarrow & \\ \mathcal{E}\text{xt}(Q, K) & \xrightarrow{B} & \text{Hom}(Q, \text{Out}(K)) \end{array}$$

commute. When K is abelian (that is when the usual epimorphism $\text{Aut}(K) \rightarrow \text{Out}(K)$ is an isomorphism), E becomes right inverse to B . Thus Γ becomes trivial (as may also be seen topologically from consideration of the universal fibration). A perhaps surprising consequence of this fact is that $\Gamma = \Gamma_K$ does *not* in general factor as

$$\text{Hom}(Q, \text{Out}(K)) \rightarrow \text{Hom}(Q, \text{Aut}(Z(K))) \xrightarrow{\Gamma_{Z(K)}} \coprod_{\alpha} H^3(Q; \{Z(K)\}_{\alpha})$$

(where $\text{Out}(K) \rightarrow \text{Aut}(Z(K))$ is induced by restriction), for the triviality of $\Gamma_{Z(K)}$ would force that of the composition Γ_K . However, after [9] (see also [11 p. 80]) one knows that for any abelian group Z and the collection \mathbf{K} of groups having Z as centre,

$$\coprod_{\mathbf{K}} \text{Hom}(Q, \text{Out}(K)) \xrightarrow{\coprod_{\Gamma_K}} \coprod_{\alpha} H^3(Q; \{Z\}_{\alpha})$$

is a surjection.

There are other favourable circumstances in which one can say a good deal further about $\mathcal{E}\text{xt}(Q, K)$. We record here two results from [3] (respectively (2.9) and (2.6)). These use the notation $\mathcal{P}G$ for the maximal perfect subgroup (perfect radical) of a group G .

PROPOSITION 1.4. *Let Q be a perfect group. If the (equivalence class of the) extension $K \rightarrow G \rightarrow Q$ lies in the image of A , then*

$$\mathcal{P}G = \mathcal{P}K \cdot \mathcal{P}C_K \cdot \mathcal{P}G(K).$$

When the kernel is hypoabelian ($\mathcal{P}K = 1$), this condition simplifies to the statement that it commutes with $\mathcal{P}G$. Here one can make explicit what additional condition is sufficient as well as necessary.

PROPOSITION 1.5. *Let Q be perfect and K hypoabelian. An extension $K \rightarrow G \xrightarrow{\pi} Q$ lies in the image of A if and only if both*

- a) π is an epimorphism preserving perfect radicals (that is, $\pi \mathcal{P}G = Q$); and
- b) $[\mathcal{P}G, K] = 1$.

These conditions are easily verified for an extension where the kernel lies in the hypercentre of G . For then K must be nilpotent, so that condition (a) is satisfied by [3 (2.3) (iii)]. On the other hand, because G acts nilpotently on K so does $\mathcal{P}G$; by Kaluzhnin's theorem [18 (7.1.1.1)] the image of the perfect group $\mathcal{P}G$ in $\text{Aut}(K)$ induced by conjugation is nilpotent and hence trivial. This result also admits a converse, for if K is nilpotent then the extension obtained by the construction A is easily seen to have its kernel in the hypercentre.

COROLLARY 1.6. *Let K be a nilpotent group and Q perfect. Then the set of equivalence classes of extensions with kernel K in the hypercentre and with quotient Q is in 1 – 1 correspondence with $H^2(Q; Z(K)) \cong \text{Hom}(H_2(Q), Z(K))$.*

Here $H_2(Q) = H_2(Q; \mathbf{Z})$ is just the Schur multiplier of Q . The given isomorphism is immediate from the universal coefficient theorem because Q is perfect.

2. RELATIVE COMPLETENESS AND CO-COMPLETENESS

This paragraph takes us to the point of departure for our examples.

PROPOSITION 2.1. *Suppose that groups Q and K have the property that every homomorphism from Q to $\text{Out}(K)$ is trivial. Then every extension with kernel K and quotient Q is trivial, provided that also either*

- (a) K is centreless, or
- (b) Q is superperfect.

Case (a) of (2.1) is of course known and includes the example of complete groups (that is, $\text{Out}(K) = 1$ too). It follows immediately from (1.3).

Case (b) requires a little more attention. Recall that Q superperfect means that its first and second homology groups with trivial integer coefficients both vanish. By means of the universal coefficient/Künneth formulae, the first and second cohomology with arbitrary trivial coefficients are also zero. So both the H^2 and Hom sets in the exact sequence of (1.1) are singletons.