1. An exact sequence

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Fixing two groups K and Q, we consider extensions G with kernel K and quotient Q. (The phraseology is intended to evade the "of K by Q" versus "of Q by K" controversy.) Strictly speaking, K is only isomorphic to the kernel, for we take an extension to be a short exact sequence of groups

$$K \overset{\iota}{\rightarrowtail} G \overset{\pi}{\twoheadrightarrow} Q\,,$$

often referring to this simply as "G".

Two extensions G, G' then are *equivalent* (also known as *congruent*) if there exists a (necessarily bijective) homomorphism $\beta: G \to G'$ making

$$\begin{tikzcd} & G \arrow[dd, "\beta"] \arrow[dr, two heads, "\pi"] & \\ K \arrow[ur, tail, "\iota"] \arrow[dr, tail, "\iota'"'] & & Q \\ & G' \arrow[ur, two heads, "\pi'"'] & \end{tikzcd}$$

commute.

The set of equivalence classes, $\mathscr{E}\mathrm{xt}(Q, K)$, is a pointed set in that it admits a distinguished element (basepoint), namely the class of the *trivial extension*

$$K \overset{in_1}{\rightarrowtail} K \times Q \overset{pr_2}{\twoheadrightarrow} Q\,.$$

It is usual either to consider more tractable subsets of this set or to specialise to the case of abelian K, so as to obtain richer algebraic structure. However here we look at $\mathscr{E}\mathrm{xt}$ in full generality. We determine it to the extent of placing this set in an exact sequence of pointed sets. (Recall that a sequence of pointed set functions

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is exact if $f(A) = g^{-1}(c_0)$, where c_0 is the basepoint of C.) For discussion of naturality of the sequence, we observe that the pointed set functor $\mathscr{E}\mathrm{xt}(\ ,\ )$ is contravariant in the quotient group via the existence of induced (pulled-back) extensions. On the other hand, in the absence of a commutativity condition it fails to be a (covariant) functor in the kernel. (More on this, later.)

PROPOSITION 1.1. *There is an exact sequence of pointed sets*

$$H^2(Q; Z(K)) \overset{A}{\rightarrowtail} \mathscr{E}\mathrm{xt}(Q, K) \xrightarrow{B} \mathrm{Hom}(Q, Out(K)) \xrightarrow{\Gamma} \coprod_{\alpha} H^3(Q; \{Z(K)\}_\alpha)$$

where the functions A, B, Γ *are defined below.*

Proof. First, an explanation of notation. $H^2(Q; Z(K))$ refers to cohomology with trivial coefficients $Z(K)$, the centre of K. On the other hand, $\{\ \}_\alpha$ indicates that the coefficients in $H^3(Q; \{Z(K)\}_\alpha)$ may be non-trivial, corresponding to a non-trivial homomorphism α from Q to the group $\mathrm{Aut}(Z(K))$ of all automorphisms of $Z(K)$. Cohomology groups, being abelian, have 0 as natural basepoint; $\coprod$ refers to the coproduct in the category of pointed sets, that is, the one-point union obtained by identifying every 0 in the disjoint union. In this case the union is taken over all possible choices of local systems of coefficients; in other words, is indexed by

$$\mathrm{Hom}(Q, \mathrm{Aut}(Z(K))) .$$

Finally, $\mathrm{Out}(K)$ denotes the outher automorphism group of K, the quotient of $\mathrm{Aut}(K)$ by its group $\mathrm{Inn}(K)$ of inner automorphisms.

Although this result may be deduced from [9] (see also [15] ch. IV, [11]), I have chosen to outline a more geometric, less ad hoc treatment here. (Equivalence of the corresponding functions occurring in the different approaches has been verified in [13].)

It is of course a standard fact (recaptured below) that $H^2(Q; Z(K))$ corresponds to the subset of $\mathscr{E}\mathrm{xt}(Q, Z(K))$ comprising central extensions. (A further topological proof, in the spirit of some of the discussion below, is presented in [2 ch. 8]. That treatment also permits a topological proof of the fact [9] that our function A generalises, to provide a bijection of each inverse image under B with the corresponding $H^2(Q; \{Z(K)\})$.)

The function A is usefully considered in somewhat fuller generality. Therefore let $\tau: Z \to L$ be a group homomorphism with domain abelian and image central in L. We define $A: H^2(Q; Z) \to \mathscr{E}\mathrm{xt}(Q, L)$ as follows. Given a central extension $Z \overset{\iota'}{\rightarrowtail} E \overset{\phi'}{\twoheadrightarrow} Q$ representing an equivalence class $[\phi'] \in H^2(Q; Z)$, let its image under A be the class of the extension

$$L \overset{\iota''}{\rightarrowtail} L \times E/\tilde{Z} \overset{\phi''}{\twoheadrightarrow} Q$$

Here the subgroup $\tilde{Z}$ of $L \times E$ consists of all pairs $(\tau(z), \iota'(z^{-1}))$, $z \in Z$, and is normal precisely because $\tau(Z)$ and $\iota'(Z)$ are both central. The homomorphisms ι'' and ϕ are the predictable ones: $\iota''(x) = (x, 1)$ and $\phi''(x, e) = \phi'(e)$. The various checks, for example that ϕ'', then A, is well-defined, are straightforward and assigned to the reader. Our proof that A is injective follows the definition of B given below. Note (for (1.2) below) that when L is abelian the resulting extension is central, so that A may be regarded as a map

$$H^2(Q; Z) \to H^2(Q; L) \hookrightarrow \mathrm{Ext}(Q, L)\,.$$

In this form, it reduces to the Baer construction, which coincides with the obvious cohomological homomorphism

$$\tau_*: H^2(Q; Z) \to H^2(Q; L)\,.$$

The function B is often referred to as the *coupling* [11 p. 65]. For a given extension $K \overset{\iota}{\rightarrowtail} G \overset{\pi}{\twoheadrightarrow} Q$ it comes from conjugation in K by inverse images in G of elements in Q. Such inverse images being determined only up to multiplication by elements of $\iota(K)$, the G-conjugation automorphism of K is defined only modulo $\mathrm{Inn}(K)$. Again, it is simple to check that B is an invariant of equivalence and thus well-defined.

Now observe that conjugation by $K \times E/\tilde{Z}$ on $\iota''(K)$ has the same effect as K-conjugation. Therefore $\mathrm{B} \circ \mathrm{A}$ is trivial. If, on the other hand, $K \overset{\iota}{\rightarrowtail} G \overset{\pi}{\twoheadrightarrow} Q$ induces trivial $Q \to \mathrm{Out}(K)$, then G coincides with the kernel $\iota K \,.\, C_G(\iota K)$ of the trivial composition of homomorphisms in the commuting square

$$\begin{array}{ccc} G & \rightarrow & \mathrm{Aut}(K) \\ {\scriptstyle \pi}\downarrow & & \downarrow \\ Q & \underset{\mathrm{B}\pi}{\rightarrow} & \mathrm{Out}(K)\,. \end{array}$$

So

$$Q \cong \iota K \,.\, C_G(\iota K)/\iota K \cong C_G(\iota K)/\iota Z(K)\,;$$

in other words, there is a central extension

$$Z(K) \overset{\iota|}{\rightarrowtail} C_G(K) \overset{\pi|}{\twoheadrightarrow} Q\,.$$

From the isomorphism

$$K \times C_G(\iota K)/\tilde{Z} \to G$$
$$(k, g) \mapsto kg$$

we infer that $A[\pi|] = [\pi]$, as required for exactness at $\mathscr{E}xt(Q, K)$. Again, if we begin with a central extension $Z(K) \overset{\iota'}{\rightarrowtail} G \overset{\phi'}{\twoheadrightarrow} Q$, then the extension $K \overset{\iota''}{\rightarrowtail} K \times E/\tilde{Z} \overset{\phi''}{\twoheadrightarrow} Q$ representing $A[\phi']$ has $Z(K) \overset{\iota''|}{\rightarrowtail} C_{K \times E/\tilde{Z}}(\iota''K) \overset{\phi''|}{\twoheadrightarrow} Q$ equivalent to ϕ'. Thus A is a bijection onto Ker B, with inverse given by restriction.

We turn now to the definition of the function Γ. At this stage classifying spaces (of topological monoids in the case of the set of self-homotopy equivalences $\mathscr{E}(X)$ and its basepoint-preserving counterpart $\mathscr{E}(X; x_0)$, otherwise of discrete groups) enter the picture. From Corollary A.5 there is a fibration

$$\mathscr{K}(Z(K), 2) \to B\mathscr{E}(X) \to BOut(K)$$

where $X = BK = \mathscr{K}(K, 1)$. A homomorphism $\psi: Q \to Out(K)$ induces $B\psi: BQ \to BOut(K)$.

$$\begin{array}{ccc} & & \mathscr{K}(Z(K), 2) \\ & & \downarrow \\ & \nearrow & B\mathscr{E}(X) \\ & & \downarrow \\ BQ & \to & BOut(K) \end{array}$$

The question as to when $B\psi$ lifts to a map $BQ \to B\mathscr{E}(X)$ (making the above triangle commute) is solved by standard obstruction theory (e.g. [23 VI (6.14)]), which asserts that there is an element of $H^3(BQ; \{Z(K)\}) = H^3(Q; \{Z(K)\})$, uniquely determined by ψ and therefore safely labelled as $\Gamma\psi$, whose vanishing is equivalent to the existence of the desired lifting. (Note that the local coefficient system $\{Z(K)\}$ is also determined by ψ via its composition with the restriction homomorphism $\mathrm{Out}(K) \to \mathrm{Aut}(Z(K))$.) Now our present claim is that $\Gamma\psi$ vanishes precisely when ψ is derived, via B, from a group extension. The link between these assertions is provided by the universality of the fibration

$$BK \to B\mathscr{E}(X; x_0) \to B\mathscr{E}(X)$$

(e.g. [7]). That is, every fibration with fibre BK is induced from this one by a map of its base space into $B\mathscr{E}(X)$. So liftings $BQ \to B\mathscr{E}(X)$ of $B\psi$ correspond one-to-one with fibrations $BK \to BG \to BQ$. (The homotopy exact sequence here shows that the total space must be a $\mathscr{K}(G, 1)$.) Finally, since the fundamental group functor is left inverse to the classifying space functor, fibrations of this form correspond one-to-one to extensions $K \rightarrowtail G \twoheadrightarrow Q$. This clinches exactness at $\mathrm{Hom}(Q, \mathrm{Out}(K))$.

In fact, the argument shows more, for it reveals that the first three terms of (1.1) are none other than those of the exact sequence

$$1 \to [BQ, \mathscr{K}(Z(K), 2)] \to [BQ, B\mathscr{E}(X)] \to [BQ, \mathrm{BOut}(K)]$$

arising from the fibration (A.5) (where the first term, $[BQ, \Omega\mathrm{BOut}(K)]$, is trivial because $\Omega\mathrm{BOut}(K) = \mathrm{Out}(K)$ is discrete). Although this does not yield that A is injective (merely that its kernel is trivial), it does provide a topological proof that $H^2(Q; Z(K)) = [BQ, \mathscr{K}(Z(K), 2)]$ maps with trivial kernel onto Ker B, which we have seen corresponds to the set of equivalence classes of central extensions with quotient Q and kernel $Z(K)$.

We now take up the matter of naturality of this sequence in the kernel K (naturality in the quotient being regarded as obvious). This has significant ramifications for us later on.

PROPOSITION 1.2. *Let* H *be a characteristic subgroup of* K. *Then the quotient homomorphism* $\kappa: K \to K/H$ *induces a map of exact sequences*

$$\begin{array}{ccccc} H^2(Q; Z(K)) & \overset{\mathrm{A}}{\rightarrowtail} & \mathscr{E}\mathrm{xt}(Q, K) & \overset{\mathrm{B}}{\to} & \mathrm{Hom}(Q, \mathrm{Out}(K)) \\ \downarrow \kappa_* & & \downarrow \kappa_* & & \downarrow \kappa_* \\ H^2(Q; Z(K/H)) & \overset{\mathrm{A}}{\rightarrowtail} & \mathscr{E}\mathrm{xt}(Q, K/H) & \overset{\mathrm{B}}{\to} & \mathrm{Hom}(Q, \mathrm{Out}(K/H)) . \end{array}$$

Moreover, if $Z(K) \leqslant H$ *and* $H/Z(K)$ *is normal when regarded as a subgroup of* $\mathrm{Aut}(K)$, *then there is a partial splitting*

$$\Delta: \mathrm{Hom}(Q, \mathrm{Out}(K)) \to \mathscr{E}\mathrm{xt}(Q, K/H)$$

such that

$$\Delta \circ \mathrm{B} = \kappa_* \quad \textit{and} \quad \mathrm{B} \circ \Delta = \kappa_* .$$

Note that the condition on $H/Z(K)$ is clearly satisfied whenever $H/Z(K)$ is characteristic in $K/Z(K) = \mathrm{Inn}(K)$, as, for example, happens when H is a member of the upper central series of K.

Proof. The cohomological map κ_* has been discussed above; its existence relies only on the normality of H in K. For the $\mathscr{E}xt$ map, let $[\pi]$ represent an extension $K \rightarrowtail G \overset{\pi}{\twoheadrightarrow} Q$. Then $\kappa_*[\pi]$ is defined to be the equivalence class of the extension $K/H \rightarrowtail G/H \twoheadrightarrow Q$. Here one needs H characteristic in K in order that H be normal in G. Also, when H is characteristic in K there is a canonical homomorphism $\hat{\kappa}\colon \mathrm{Out}(K) \to \mathrm{Out}(K/H)$. So $\kappa_*\colon \mathrm{Hom}(Q, \mathrm{Out}(K)) \to \mathrm{Hom}(Q, \mathrm{Out}(K/H))$ is given simply by composition with $\hat{\kappa}$. Verification of commutativity of the two squares is a tedious but uncomplicated exercise.

The map Δ is more interesting. It can be viewed as the composition of two of the three constructions on extensions already presented. Beginning with the standard extension

$$K/Z(K) = \mathrm{Inn}(K) \rightarrowtail \mathrm{Aut}(K) \twoheadrightarrow \mathrm{Out}(K),$$

we obtain by assumption a second extension

$$K/H \rightarrowtail \mathrm{Aut}(K)\,/\,\big(H/Z(K)\big) \twoheadrightarrow \mathrm{Out}(K),$$

and then pull it back over a given homomorphism $\psi\colon Q \to \mathrm{Out}(K)$ to obtain as $\Delta(\psi)$ the induced extension with quotient Q and kernel K/H. The check of commutativity of the two triangles formed is again routine. (In the case $H = Z(K)$, Rose [20] calls the values of Δ *sited extensions.*)

One is tempted to speculate on the existence of a map κ_* at the H^3 level. However this first requires a map of coefficient systems. There is in general no function $\mathrm{Aut}(Z(K)) \to \mathrm{Aut}(Z(K/H))$ such that the square

$$\begin{array}{ccc} \mathrm{Out}(K) & \to & \mathrm{Aut}(Z(K)) \\ \downarrow & & \downarrow \\ \mathrm{Out}(K/H) & \to & \mathrm{Aut}(Z(K/H)) \end{array}$$

(whose horizontal maps are given by restriction) commutes, as may be seen by reference to the example where K is the centreless alternating group A_4 and H is the four-group, a characteristic subgroup. For then $\mathrm{Aut}(Z(K))$ is trivial, while

$$\mathrm{Out}(K) \to \mathrm{Out}(K/H) \cong \mathrm{Aut}(Z(K/H))$$

is surjective and non-trivial (see, for example, [22]).

An immediate consequence of (1.2) is familiar (for example [20]).

COROLLARY 1.3. *If* $Z(K) = 1$, *then* B *and* Δ *are inverse bijections.*

In particular, every extension with kernel K *is induced from the extension* $K \rightarrowtail \mathrm{Aut}(K) \twoheadrightarrow \mathrm{Out}(K)$ *by a homomorphism into* $\mathrm{Out}(K)$.

Another sense in which B admits a partial inverse is provided by the semi-direct product construction (described in for example [21 Theorem 9.9]). This may be regarded as an injection $\mathrm{E}: \mathrm{Hom}(Q, \mathrm{Aut}(K)) \rightarrowtail \mathscr{E}\mathrm{xt}(Q, K)$, which evidently makes the triangle

$$\begin{array}{ccc} & \mathrm{E} \swarrow & \mathrm{Hom}(Q, \mathrm{Aut}(K)) \\ & & \downarrow \\ \mathscr{E}\mathrm{xt}(Q, K) & \xrightarrow{\mathrm{B}} & \mathrm{Hom}(Q, \mathrm{Out}(K)) \end{array}$$

commute. When K is abelian (that is when the usual epimorphism $\mathrm{Aut}(K) \to \mathrm{Out}(K)$ is an isomorphism), E becomes right inverse to B. Thus Γ becomes trivial (as may also be seen topologically from consideration of the universal fibration). A perhaps surprising consequence of this fact is that $\Gamma = \Gamma_K$ does *not* in general factor as

$$\mathrm{Hom}(Q, \mathrm{Out}(K)) \to \mathrm{Hom}(Q, \mathrm{Aut}(Z(K))) \xrightarrow{\Gamma_{Z(K)}} \coprod_{\alpha} H^3(Q; \{Z(K)\}_\alpha)$$

(where $\mathrm{Out}(K) \to \mathrm{Aut}(Z(K))$ is induced by restriction), for the triviality of $\Gamma_{Z(K)}$ would force that of the composition Γ_K. However, after [9] (see also [11 p. 80]) one knows that for any abelian group Z and the collection $\mathbf{K}$ of groups having Z as centre,

$$\coprod_{\mathbf{K}} \mathrm{Hom}(Q, \mathrm{Out}(K)) \xrightarrow{\coprod \Gamma_K} \coprod_{\alpha} H^3(Q; \{Z\}_\alpha)$$

is a surjection.

There are other favourable circumstances in which one can say a good deal further about $\mathscr{E}\mathrm{xt}(Q, K)$. We record here two results from [3] (respectively (2.9) and (2.6)). These use the notation $\mathscr{P}G$ for the maximal perfect subgroup (perfect radical) of a group G.

PROPOSITION 1.4. *Let* Q *be a perfect group. If the (equivalence class of the) extension* $K \rightarrowtail G \twoheadrightarrow Q$ *lies in the image of* A, *then*

$$\mathscr{P}G = \mathscr{P}K \,.\, \mathscr{P}C_{K.\mathscr{P}G}(K)\,.$$

When the kernel is hypoabelian ($\mathscr{P}K = 1$), this condition simplifies to the statement that it commutes with $\mathscr{P}G$. Here one can make explicit what additional condition is sufficient as well as necessary.

PROPOSITION 1.5. *Let* Q *be perfect and* K *hypoabelian. An extension* $K \rightarrowtail G \overset{\pi}{\twoheadrightarrow} Q$ *lies in the image of* A *if and only if both*

a) π *is an epimorphism preserving perfect radicals (that is,* $\pi\mathscr{P}G = Q$*); and*

b) $[\mathscr{P}G, K] = 1$.

These conditions are easily verified for an extension where the kernel lies in the hypercentre of G. For then K must be nilpotent, so that condition (a) is satisfied by [3 (2.3) (iii)]. On the other hand, because G acts nilpotently on K so does $\mathscr{P}G$; by Kaluzhnin's theorem [18 (7.1.1.1)] the image of the perfect group $\mathscr{P}G$ in Aut(K) induced by conjugation is nilpotent and hence trivial. This result also admits a converse, for if K is nilpotent then the extension obtained by the construction A is easily seen to have its kernel in the hypercentre.

COROLLARY 1.6. *Let* K *be a nilpotent group and* Q *perfect. Then the set of equivalence classes of extensions with kernel* K *in the hypercentre and with quotient* Q *is in* $1-1$ *correspondence with* $H^2(Q; Z(K)) \cong \mathrm{Hom}(H_2(Q), Z(K))$.

Here $H_2(Q) = H_2(Q; \mathbf{Z})$ is just the Schur multiplier of Q. The given isomorphism is immediate from the universal coefficient theorem because Q is perfect.

2. RELATIVE COMPLETENESS AND CO-COMPLETENESS

This paragraph takes us to the point of departure for our examples.

PROPOSITION 2.1. *Suppose that groups* Q *and* K *have the property that every homomorphism from* Q *to* Out(K) *is trivial. Then every extension with kernel* K *and quotient* Q *is trivial, provided that also either*

(a) K *is centreless, or*

(b) Q *is superperfect.*

Case (a) of (2.1) is of course known and includes the example of complete groups (that is, Out(K)$=1$ too). It follows immediately from (1.3).

Case (b) requires a little more attention. Recall that Q superperfect means that its first and second homology groups with trivial integer coefficients both vanish. By means of the universal coefficient/Künneth formulae, the first and second cohomology with arbitrary trivial coefficients are also zero. So both the H^2 and Hom sets in the exact sequence of (1.1) are singletons.