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A HOLOMORPHICALLY SEPARABLE COMPLEX SPACE WITHOUT THE GELFAND TOPOLOGY

by Sandra HAYES-WIDMANN

ABSTRACT

An example of a holomorphically separable complex space with a Stein envelope of holomorphy which does not carry the Gelfand topology is given. This example also shows that an injective holomorphic map $\varphi\colon X \to Y$ between complex spaces with $\dim_x X = \dim_{\varphi(x)} Y$, $x \in X$, is not always open, even when φ is the canonical map of a pre-Stein space X into its envelope of holomorphy.

INTRODUCTION

The Gelfand topology for a reduced complex space $(X, \mathcal{O})$ is the weak topology on X determined by the global function algebra $\mathcal{O}(X)$. Since only holomorphically separable complex spaces can carry this topology, it is natural to ask whether holomorphic separability characterizes those complex spaces with the Gelfand topology. A remark in [4, Bemerkung 3] implies that this is the case, at least for pre-Stein spaces. However, a counter-example given here shows that holomorphically separable spaces need not have the Gelfand topology, even when they are pre-Stein.

EXAMPLE

If a complex space $(X, \mathcal{O})$ is furnished with the Gelfand topology, then it must be holomorphically separable in a strong sense—every interior point can be separated not only from every other interior point but also from every "boundary" point by a global holomorphic function. More precisely, the latter separation property means that for every point $x \in X$ and for every sequence $(x_n)_{n \in \mathbf{N}}$ in $X \setminus \{x\}$ with no cluster point in X, there exists a global holomorphic function $f \in \mathcal{O}(X)$ such that

$$f(x) \notin \overline{\{ f(x_n) \mid n \in \mathbf{N} \}} .$$

The following example shows that holomorphically separable complex spaces having interior points which cannot be separated from boundary points actually exist. I am indebted to J. P. Vigué for the construction involved in this example.

In $\mathbf{C}^3$ with the coordinates x, y, z denote by

$$C := \left\{ (0, y, 0) \in \mathbf{C}^3 \;\middle|\; |y| \leqslant \frac{1}{2} \right\}$$

the circle with radius 1/2 around the origin in the y-plane. In $\{0\} \, x \, \mathbf{C}^2$ let

$$Y := \left\{ (0, y, z) \in \mathbf{C}^3 \;\middle|\; |y| < 1, |z| < 1 \right\} \setminus C$$

be the unit bidisc with C omitted. Let Z be the unit bidisc in $\mathbf{C}^2 \, x \, \{0\}$ with the circumference of C elected, i.e.

$$Z := \left\{ (x, y, 0) \in \mathbf{C}^3 \;\middle|\; |x| < 1, |y| < 1 \right\} \setminus \left\{ (0, y, 0) \in \mathbf{C}^3 \;\middle|\; |y| = \frac{1}{2} \right\} .$$

The ring

$$R := \left\{ (0, y, 0) \in \mathbf{C}^3 \;\middle|\; \frac{1}{2} < |y| < 1 \right\} .$$

is an analytic subset of X as well as of Y. Attach Y to Z along R and call the resulting space X. This space, which is the fiber sum (pushout) $Y +_R Z$ of Y and Z under the inclusions $R \to Y$ and $R \to Z$, is a holomorphically separable complex space [2].

X cannot have the Gelfand topology. To see this, observe that X is the disjoint union of Y and Z with points of R identified. Consequently, the origin in $\mathbf{C}^3$ can be considered as an interior point in X, since it is a point in Z, as well as a point not belonging to X, since it isn't in Y. These two points can't be separated by any global holomorphic function.

X is a pre-Stein space, i.e. X has a Stein envelope of holomorphy as defined in [1]. This follows from the fact that $\mathcal{O}(X)$ is the algebra $\mathcal{O}(D)$ of global holomorphic functions on the Stein space $D := D_1 +_E D_2$ obtained by attaching the unit bidisc D_1 in $\{0\} x \mathbf{C}^2$ to the unit bidisc D_2 in $\mathrm{C}_2 x \{0\}$ along

$$E := \left\{ (0, y, 0) \in \mathbf{C}^3 \;\middle|\; |y| < 1 \right\}.$$

It is well known that every Stein space is equipped with the Gelfand topology (see [1]).

There is a classical dimension formula [3, Sätze 28, 29] for an injective holomorphic map $\varphi : X \to Y$ between complex spaces where Y is locally irreducible which states that φ is open, if $\dim_x X = \dim_{\varphi(x)} Y$ for $x \in X$. According to the above example, this formula cannot be generalized to maps $\varphi : X \to Y$ if Y is not locally irreducible, not even when Y is the Stein envelope of holomorphy of X and φ is the canonical map which takes points x of X to the corresponding point evaluations $\mathcal{O}(X) \mapsto \mathbf{C}$, $f \to f(x)$, in the continuous spectrum $S_c(\mathcal{O}(X))$.

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