

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 31 (1985)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: HOLOMORPHICALLY SEPARABLE COMPLEX SPACE WITHOUT THE GELFAND TOPOLOGY
Autor: Hayes-Widmann, Sandra
Kapitel: Example
DOI: <https://doi.org/10.5169/seals-54562>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 02.09.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

A HOLOMORPHICALLY SEPARABLE COMPLEX SPACE WITHOUT THE GELFAND TOPOLOGY

by Sandra HAYES-WIDMANN

ABSTRACT

An example of a holomorphically separable complex space with a Stein envelope of holomorphy which does not carry the Gelfand topology is given. This example also shows that an injective holomorphic map $\varphi: X \rightarrow Y$ between complex spaces with $\dim_x X = \dim_{\varphi(x)} Y$, $x \in X$, is not always open, even when φ is the canonical map of a pre-Stein space X into its envelope of holomorphy.

INTRODUCTION

The Gelfand topology for a reduced complex space (X, \mathcal{O}) is the weak topology on X determined by the global function algebra $\mathcal{O}(X)$. Since only holomorphically separable complex spaces can carry this topology, it is natural to ask whether holomorphic separability characterizes those complex spaces with the Gelfand topology. A remark in [4, Bemerkung 3] implies that this is the case, at least for pre-Stein spaces. However, a counter-example given here shows that holomorphically separable spaces need not have the Gelfand topology, even when they are pre-Stein.

EXAMPLE

If a complex space (X, \mathcal{O}) is furnished with the Gelfand topology, then it must be holomorphically separable in a strong sense—every interior point can be separated not only from every other interior point but also from every “boundary” point by a global holomorphic function. More precisely,

the latter separation property means that for every point $x \in X$ and for every sequence $(x_n)_{n \in \mathbb{N}}$ in $X \setminus \{x\}$ with no cluster point in X , there exists a global holomorphic function $f \in \mathcal{O}(X)$ such that

$$f(x) \notin \overline{\{f(x_n) \mid n \in \mathbb{N}\}}.$$

The following example shows that holomorphically separable complex spaces having interior points which cannot be separated from boundary points actually exist. I am indebted to J. P. Vigué for the construction involved in this example.

In \mathbf{C}^3 with the coordinates x, y, z denote by

$$C := \left\{ (0, y, 0) \in \mathbf{C}^3 \mid |y| \leq \frac{1}{2} \right\}$$

the circle with radius $1/2$ around the origin in the y -plane. In $\{0\} \times \mathbf{C}^2$ let

$$Y := \left\{ (0, y, z) \in \mathbf{C}^3 \mid |y| < 1, |z| < 1 \right\} \setminus C$$

be the unit bidisc with C omitted. Let Z be the unit bidisc in $\mathbf{C}^2 \times \{0\}$ with the circumference of C elected, i.e.

$$Z := \left\{ (x, y, 0) \in \mathbf{C}^3 \mid |x| < 1, |y| < 1 \right\} \setminus \left\{ (0, y, 0) \in \mathbf{C}^3 \mid |y| = \frac{1}{2} \right\}.$$

The ring

$$R := \left\{ (0, y, 0) \in \mathbf{C}^3 \mid \frac{1}{2} < |y| < 1 \right\}$$

is an analytic subset of X as well as of Y . Attach Y to Z along R and call the resulting space X . This space, which is the fiber sum (pushout) $Y +_R Z$ of Y and Z under the inclusions $R \rightarrow Y$ and $R \rightarrow Z$, is a holomorphically separable complex space [2].

X cannot have the Gelfand topology. To see this, observe that X is the disjoint union of Y and Z with points of R identified. Consequently, the origin in \mathbf{C}^3 can be considered as an interior point in X , since it is a point in Z , as well as a point not belonging to X , since it isn't in Y . These two points can't be separated by any global holomorphic function.

X is a pre-Stein space, i.e. X has a Stein envelope of holomorphy as defined in [1]. This follows from the fact that $\mathcal{O}(X)$ is the algebra $\mathcal{O}(D)$ of global holomorphic functions on the Stein space $D := D_1 + {}_E D_2$ obtained by attaching the unit bidisc D_1 in $\{0\} \times \mathbf{C}^2$ to the unit bidisc D_2 in $\mathbf{C}_2 \times \{0\}$ along

$$E := \left\{ (0, y, 0) \in \mathbf{C}^3 \mid |y| < 1 \right\}.$$

It is well known that every Stein space is equipped with the Gelfand topology (see [1]).

There is a classical dimension formula [3, Sätze 28, 29] for an injective holomorphic map $\varphi: X \rightarrow Y$ between complex spaces where Y is locally irreducible which states that φ is open, if $\dim_x X = \dim_{\varphi(x)} Y$ for $x \in X$. According to the above example, this formula cannot be generalized to maps $\varphi: X \rightarrow Y$ if Y is not locally irreducible, not even when Y is the Stein envelope of holomorphy of X and φ is the canonical map which takes points x of X to the corresponding point evaluations $\mathcal{O}(X) \mapsto \mathbf{C}, f \mapsto f(x)$, in the continuous spectrum $S_c(\mathcal{O}(X))$.

REFERENCES

- [1] FORSTER, O. Holomorphegebieten. In: H. Behnke and P. Thullen, *Theorie der Funktionen mehrerer komplexen Veränderlichen*, 2. edition, Springer, Berlin 1970 (Anhang zu VI).
- [2] KAUP, B. Ueber Kokerne und Pushouts in der Kategorie der komplex-analytischen Räume. *Math. Ann.* 189 (1970), 60-76.
- [3] REMMERT, R. Holomorphe und meromorphe Abbildungen komplexer Räume. *Math. Ann.* 133 (1957), 328-370.
- [4] WIEGMANN, K. W. Ein holomorph-separabler komplexer Raum muss nicht holomorph-regulär sein. *L'Enseignement Math.* 14 (1968), 283-284.

(Reçu le 5 août 1984)

Sandra Hayes-Widmann

Mathematisches Institut der technischen Universität
Postfach 20 24 20
D-8000 München