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4.2. LEMMA. Let L be a lattice in \mathbf{R}^n whose elements have pairwise distances ≥ 1 and consider a linear subspace E of \mathbf{R}^n which is spanned by k vectors $w_1, \dots, w_k \in L$. If a lattice point $w \in L$ is not contained in E , then its E^\perp -component w^\perp has length

$$|w^\perp| \geq (3 + |w_1| + \dots + |w_k|)^{-n}.$$

Proof. Let N be the integer part of $(3 + |w_1| + \dots + |w_k|)^n$. If $0 < |w^\perp| \leq 1/N$, then $0, w, 2w, \dots, Nw$ have distance ≤ 1 from E . Adding suitable integer linear combinations of w_1, \dots, w_k to each of these vectors we obtain $N + 1$ new pairwise different lattice points whose E^\perp components have not changed but whose E components are $\leq \frac{1}{2}(|w_1| + \dots + |w_k|)$. These $N + 1$ lattice points have distance $\leq 1 + \frac{1}{2}(|w_1| + \dots + |w_k|)$ from the origin, a contradiction to Lemma 4.1.

5. PROOF OF THEOREM II

For an n -dimensional crystallographic group G we let $L(G)$ be the subgroup consisting of all pure translations in G . By Theorem I, $L(G)$ is a lattice in \mathbf{R}^n . The standard observation which is "responsible" for Theorem II is

5.1. LEMMA. If $\alpha \in G$ and if $w \in L(G)$, then $A(w) \in L(G)$, ($A = \text{rot } \alpha$).

Proof. Recall that $w = \text{trans } \omega$, $\omega \in L(G)$. Now $\alpha\omega\alpha^{-1} \in G$ is a translation with translation vector $A(w)$. Hence $A(w) \in L(G)$.

5.2. Definition. A crystallographic group is called *normal* if

- (i) the vectors in $L(G)$ have pairwise distances ≥ 1
- (ii) $L(G)$ contains n linearly independent unit vectors.

We do not ask that the vectors in (ii) generate the entire lattice $L(G)$.

Our idea is to count the normal groups. This will suffice due to the following.

5.3. PROPOSITION. Each crystallographic group G is isomorphic to a normal crystallographic group.

Proof. By scaling we may assume that the shortest non zero vector in $L(G)$ is a unit vector. Now assume by induction that $L(G)$ satisfies 5.2 (i) and contains $k < n$ unit vectors w_1, \dots, w_k which span a k -dimensional linear subspace E of \mathbf{R}^n . It remains to find a group G' isomorphic to G such that $L(G)$ contains $k + 1$ linearly independent unit vectors and also satisfies 5.2 (i).

If for some $\alpha \in G$ and for some $w_i (i \leq k)$ the vector $A(w_i)$ is not contained in E , then by Lemma 5.1 $A(w_i) \in L(G)$ is already the $(k + 1)$ -st vector and we are done.

If on the other hand all rotation parts of G leave E —and consequently E^\perp —invariant, then the affine transformations Φ_μ given by

$$\Phi_\mu(x^E + x^\perp) = x^E + \mu x^\perp$$

($\mu > 0$) commute with the rotation parts of G . Therefore, the affine conjugate (and henceforth isomorphic) groups $G_\mu = \Phi_\mu G \Phi_\mu^{-1}$ also act by rigid motions. Since $L(G_\mu) = \Phi_\mu(L(G))$, Lemma 4.2 implies that G_μ violates 5.2 (i) if $\mu > 0$ is very small. Hence there exists a *minimal* $\mu' > 0$ such that $G_{\mu'}$ satisfies 5.2 (i). Since the affine transformations Φ_μ act trivially on E , the shortest vector in $L(G_{\mu'}) \setminus E$ must be a unit vector and $w_1, \dots, w_k \in L(G_{\mu'})$. Now $G_{\mu'}$ has the required properties and Proposition 5.3 is proved.

5.4. *The proof of Theorem II now proceeds in two steps.*

Step 1. Each normal crystallographic group G is uniquely characterized by a group table ((ii) below).

Proof. Fix n linearly independent unit vectors $w_1, \dots, w_n \in L(G)$ and consider the sublattice

$$L = \{m_1 w_1 + \dots + m_n w_n \mid m_1, \dots, m_n \in \mathbf{Z}\}.$$

L is a subgroup of G . In each right coset modulo L of G we select a representative ω whose translation part w has length

$$(i) \quad |w| \leq \frac{1}{2} (|w_1| + \dots + |w_n|) = \frac{n}{2}.$$

Since G is discrete (2.4. (i)), there are only finitely many such representatives, say $\omega_{n+1}, \dots, \omega_N$. Every $\alpha \in G$ can now be expressed in a unique way in the form

$$\alpha = (m_1 w_1 + \dots + m_n w_n) \omega_\nu$$

where $n + 1 \leq v \leq N$. Since our L is isomorphic to \mathbf{Z}^n , G is uniquely determined (up to isomorphism) by the integers m_{ijk} , $v(j, k)$ and N which occur in the table

$$(ii) \quad \omega_j \omega_k = (m_{1jk} w_1 + \dots + m_{njk} w_n) \omega_{v(j, k)}, \quad j, k = 1, \dots, N.$$

(For $i = 1, \dots, n$, ω_i is the translation by w_i).

Clearly, the proof of Theorem II will be completed by

Step 2. The absolute values of m_{ijk} , $v(j, k)$ and N in (ii) have an upper bound which depends only on the dimension n (see (iii) and (iv) below).

Proof. The euclidean motions $\omega_{v(j, k)}$, ω_j and ω_k in (ii) have translation parts of length $\leq \frac{n}{2}$ (c.f. (i)). Consequently the translation $m_{1jk} w_1 + \dots + m_{njk} w_n = \omega_j \omega_k \omega_{v(j, k)}^{-1}$ has length $\leq \frac{3n}{2}$. In particular,

$$|m_{ijk} w_i^\perp| \leq \frac{3n}{2}, \quad i = 1, \dots, n$$

where w_i^\perp is the component of w_i perpendicular to the hyperplane E spanned by $w_1, \dots, w_{i-1}, w_{i+1}, \dots, w_n$. By Lemma 4.2 we have $|w_i^\perp| \geq (n+2)^{-n}$. Hence

$$(iii) \quad |m_{ijk}| \leq \frac{3n}{2} (n+2)^n.$$

Now let us estimate N . The linear transformation $A = \text{rot } \alpha$, $\alpha \in G$ is uniquely determined by its images $A(w_i)$, $i = 1, \dots, n$. By Lemma 5.1 each of these images is a unit vector of $L(G)$ and, by Lemma 4.1, one out of at most 3^n candidates. It follows that at most $(3^n)^n$ different rotation parts occur in G .

If two elements ω_ρ and ω_σ among $\omega_{n+1}, \dots, \omega_N$ have the same rotation part, then $\omega_\rho \omega_\sigma^{-1}$ is a vector of length $\leq \frac{n}{2} + \frac{n}{2}$ (c.f. (i)) and, again by Lemma 4.1, one out of at most $(2n+1)^n$ candidates. Hence

$$(iv) \quad N \leq n + (3^n)^n \cdot (2n+1)^n.$$

Since $v(i, j) \leq N$, this concludes the proof of Theorem II.

5.5. *Remark.* From the preceding proof we can derive the upper bound $\exp \exp 4n^2$ for the number of isomorphism classes of n -dimensional crystallographic groups. The correct numbers for $n = 1, 2, 3, 4$ are respectively 2, 17, 219, 4783 [4].

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