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$$|r| < \frac{1}{2}\left(m(A) - m^{\perp}(A)\right) |b^E| .$$

Together with 2.2 (iv) we obtain

$$|\tilde{b}^{\perp}| < \frac{1}{2}\left(m(A) + m^{\perp}(A)\right) |b^E| < |\tilde{b}^E| .$$

We find that $\tilde{\beta}$ also satisfies (*), but with $|\tilde{b}| \leqslant m(A)|b| - r < |b|$, a contradiction.

3.3. *End of proof.* Lemma A provides elements in G with n linearly independent translation parts whose rotation parts are smaller than $\frac{1}{2}$. By Lemma B these elements are pure translations.

4. Lattices

In this paragraph we collect the rudiments from lattice point theory which are necessary for the proof of Theorem II. A lattice L is a crystallographic group which consists only of translations. The elements of L (lattice points) will be identified with vectors in $\mathbf{R}^n$. By abuse of notation, we shall write $\omega = w = \text{trans}\,\omega$ for $\omega \in L$. It is well known that L is isomorphic to $\mathbf{Z}^n$ but this fact will *not* be used in our proof of Theorem II. Notice, however that L is abelian and that the minimal distance of lattice points equals the length of the smallest non-zero element in L.

4.1. Lemma. *Let L be a lattice in $\mathbf{R}^n$ whose elements have pairwise distances $\geqslant 1$, and let $N(\rho)$ denote the number of lattice points in L whose distance from the origin is $\leqslant \rho$ $(\rho > 0)$. Then*

$$N(\rho) \leqslant (2\rho + 1)^n .$$

Proof. The open balls of radius $\frac{1}{2}$ around the $N(\rho)$ lattice points are pairwise disjoint and all contained in a ball of radius $\rho + \frac{1}{2}$. Comparing the volumes we find $N(\rho)\left(\frac{1}{2}\right)^n \leqslant \left(\rho + \frac{1}{2}\right)^n$.

4.2. LEMMA. *Let L be a lattice in $\mathbf{R}^n$ whose elements have pairwise distances $\geqslant 1$ and consider a linear subspace E of $\mathbf{R}^n$ which is spanned by k vectors $w_1, \ldots, w_k \in L$. If a lattice point $w \in L$ is not contained in E, then its $E^{\perp}$-component $w^{\perp}$ has length*

$$| w^{\perp} | \geqslant (3+|w_1|+\ldots+|w_k|)^{-n} .$$

Proof. Let N be the integer part of $(3+|w_1|+\ldots+|w_k|)^n$. If $0 < | w^{\perp} | \leqslant 1/N$, then $0, w, 2w, \ldots, Nw$ have distance $\leqslant 1$ from E. Adding suitable integer linear combinations of $w_1, \ldots, w_k$ to each of these vectors we obtain $N+1$ new pairwise different lattice points whose $E^{\perp}$ components have not changed but whose E components are $\leqslant \frac{1}{2}(|w_1|+\ldots+|w_k|)$. These $N+1$ lattice points have distance $\leqslant 1 + \frac{1}{2}(|w_1|+\ldots+|w_k|)$ from the origin, a contradiction to Lemma 4.1.

5. PROOF OF THEOREM II

For an n-dimensional crystallographic group G we let $L(G)$ be the subgroup consisting of all pure translations in G. By Theorem I, $L(G)$ is a lattice in $\mathbf{R}^n$. The standard observation which is "responsible" for Theorem II is

5.1. LEMMA. *If $\alpha \in G$ and if $w \in L(G)$, then $A(w) \in L(G)$, $(A = \text{rot}\, \alpha)$.*

Proof. Recall that $w = \text{trans}\, \omega$, $\omega \in L(G)$. Now $\alpha\omega\alpha^{-1} \in G$ is a translation with translation vector $A(w)$. Hence $A(w) \in L(G)$.

5.2. *Definition.* A crystallographic group is called *normal* if

(i) the vectors in $L(G)$ have pairwise distances $\geqslant 1$

(ii) $L(G)$ contains n linearly independent *unit* vectors.

We do not ask that the vectors in (ii) generate the entire lattice $L(G)$.

Our idea is to count the normal groups. This will suffice due to the following.

5.3. PROPOSITION. *Each crystallographic group G is isomorphic to a normal crystallographic group.*