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$$|r| < \frac{1}{2}(m(A) - m^\perp(A)) |b^E|.$$

Together with 2.2 (iv) we obtain

$$|\tilde{b}^\perp| < \frac{1}{2}(m(A) + m^\perp(A)) |b^E| < |\tilde{b}^E|.$$

We find that $\tilde{\beta}$ also satisfies (*), but with $|\tilde{b}| \leq m(A)|b| - r < |b|$, a contradiction.

3.3. *End of proof.* Lemma A provides elements in G with n linearly independent translation parts whose rotation parts are smaller than $\frac{1}{2}$. By Lemma B these elements are pure translations.

4. LATTICES

In this paragraph we collect the rudiments from lattice point theory which are necessary for the proof of Theorem II. A lattice L is a crystallographic group which consists only of translations. The elements of L (lattice points) will be identified with vectors in \mathbf{R}^n . By abuse of notation, we shall write $\omega = w = \text{trans } \omega$ for $\omega \in L$. It is well known that L is isomorphic to \mathbf{Z}^n but this fact will *not* be used in our proof of Theorem II. Notice, however that L is abelian and that the minimal distance of lattice points equals the length of the smallest non-zero element in L .

4.1. LEMMA. *Let L be a lattice in \mathbf{R}^n whose elements have pairwise distances ≥ 1 , and let $N(\rho)$ denote the number of lattice points in L whose distance from the origin is $\leq \rho$ ($\rho > 0$). Then*

$$N(\rho) \leq (2\rho + 1)^n.$$

Proof. The open balls of radius $\frac{1}{2}$ around the $N(\rho)$ lattice points are pairwise disjoint and all contained in a ball of radius $\rho + \frac{1}{2}$. Comparing the volumes we find $N(\rho) \left(\frac{1}{2}\right)^n \leq \left(\rho + \frac{1}{2}\right)^n$.

4.2. LEMMA. Let L be a lattice in \mathbf{R}^n whose elements have pairwise distances ≥ 1 and consider a linear subspace E of \mathbf{R}^n which is spanned by k vectors $w_1, \dots, w_k \in L$. If a lattice point $w \in L$ is not contained in E , then its E^\perp -component w^\perp has length

$$|w^\perp| \geq (3 + |w_1| + \dots + |w_k|)^{-n}.$$

Proof. Let N be the integer part of $(3 + |w_1| + \dots + |w_k|)^n$. If $0 < |w^\perp| \leq 1/N$, then $0, w, 2w, \dots, Nw$ have distance ≤ 1 from E . Adding suitable integer linear combinations of w_1, \dots, w_k to each of these vectors we obtain $N + 1$ new pairwise different lattice points whose E^\perp components have not changed but whose E components are $\leq \frac{1}{2}(|w_1| + \dots + |w_k|)$. These $N + 1$ lattice points have distance $\leq 1 + \frac{1}{2}(|w_1| + \dots + |w_k|)$ from the origin, a contradiction to Lemma 4.1.

5. PROOF OF THEOREM II

For an n -dimensional crystallographic group G we let $L(G)$ be the subgroup consisting of all pure translations in G . By Theorem I, $L(G)$ is a lattice in \mathbf{R}^n . The standard observation which is "responsible" for Theorem II is

5.1. LEMMA. If $\alpha \in G$ and if $w \in L(G)$, then $A(w) \in L(G)$, ($A = \text{rot } \alpha$).

Proof. Recall that $w = \text{trans } \omega$, $\omega \in L(G)$. Now $\alpha\omega\alpha^{-1} \in G$ is a translation with translation vector $A(w)$. Hence $A(w) \in L(G)$.

5.2. Definition. A crystallographic group is called *normal* if

- (i) the vectors in $L(G)$ have pairwise distances ≥ 1
- (ii) $L(G)$ contains n linearly independent unit vectors.

We do not ask that the vectors in (ii) generate the entire lattice $L(G)$.

Our idea is to count the normal groups. This will suffice due to the following.

5.3. PROPOSITION. Each crystallographic group G is isomorphic to a normal crystallographic group.