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Since A is an orthogonal map, the orthogonal complement E_A^\perp of E_A is also an A -invariant linear subspace of \mathbf{R}^n . We define

$$(ii) \quad m^\perp(A) = \max \{ |Ax - x| / |x| \mid x \in E_A^\perp \setminus \{0\} \}$$

if $E_A^\perp \neq \{0\}$ and set $m^\perp(A) = 0$ if $E_A^\perp = \{0\}$. It follows that

$$(iii) \quad m^\perp(A) < m(A) \text{ if } A \neq id.$$

We let $x = x^E + x^\perp$, $x^E \in E_A$, $x^\perp \in E_A^\perp$ be the orthogonal decomposition of a vector x with respect to E_A and E_A^\perp . The simple observation

$$(iv) \quad |Ax^E - x^E| = m(A) |x^E|, \quad |Ax^\perp - x^\perp| \leq m^\perp(A) |x^\perp|$$

together with (iii), will play a crucial role in the proof of Theorem I.

2.3. *Commutator estimate.* For $A, B \in O(n)$ we have

$$m([A, B]) \leq 2m(A) m(B).$$

Proof. Verify the identity

$$[A, B] - id = ((A - id)(B - id) - (B - id)(A - id))A^{-1}B^{-1}$$

From $|A^{-1}B^{-1}x| = |x|$ it then follows that

$$|[A, B]x - x| \leq m(A) m(B) |x| + m(B) m(A) |x|$$

for all $x \in \mathbf{R}^n$.

2.4. *Crystallographic groups.* Discreteness and compactness of the fundamental domain will be used as follows:

A group G of rigid motions in \mathbf{R}^n is called crystallographic if

- (i) for all $t > 0$ only finitely many $\alpha \in G$ have $|a| \leq t$,
- (ii) there is some constant d such that for each $x \in \mathbf{R}^n$ there is an element $\alpha \in G$ satisfying $|a - x| \leq d$.

3. PROOF OF THEOREM I

Now let G be an n -dimensional crystallographic group.

3.1. LEMMA A ("Mini Bieberbach"). For each unit vector $u \in \mathbf{R}^n$ and for all $\varepsilon, \delta > 0$ there exists $\beta \in G$ satisfying

$$b \neq 0, \quad \angle(u, b) \leq \delta, \quad m(B) \leq \varepsilon.$$

Proof. By 2.4 (ii) there exists an element $\beta_k \in G$ satisfying $|b_k - k u| \leq d$, for each $k = 1, 2, \dots$. The sequence β_1, β_2, \dots satisfies

$$|b_k| \rightarrow \infty, \quad \angle(u, b_k) \rightarrow 0 \quad (k \rightarrow \infty).$$

Since $O(n)$ is compact, we find a subsequence such that the rotation parts B_k also converge. Consequently there exist $i < j$ such that

$$m(B_j B_i^{-1}) \leq \varepsilon, \quad \angle(u, b_j) \leq \delta/2, \quad |b_i| \leq \frac{\delta}{4} |b_j|.$$

The motion $x \mapsto \beta_j \beta_i^{-1} x = B_j B_i^{-1} x + b_j - B_j B_i^{-1} b_i$ has now all the required properties.

3.2. LEMMA B. *If $\alpha \in G$ satisfies $m(A) \leq \frac{1}{2}$, then α is a pure translation.*

Proof. If G contains elements α satisfying $0 < m(A) \leq \frac{1}{2}$, we consider the one for which $|a|$ is a *minimum* (2.4 (i)). Lemma A (applied to an arbitrary unit vector $u \in E_A$) provides elements $\beta \in G$ satisfying

$$(*) \quad b \neq 0, \quad |b^\perp| \leq |b^E|, \quad m(B) \leq \frac{1}{8} (m(A) - m^\perp(A))$$

(c.f. 2.2. (iii)). Among these we again consider the one for which $|b|$ is a minimum ($\neq 0!$). Observe that $|b| \geq |a|$ if β is not a translation by the choice of α .

The commutator $\tilde{\beta} = [\alpha, \beta]$ satisfies

$$m(\tilde{B}) = m([A, B]) \leq 2m(A) m(B) \leq m(B)$$

(2.3), and we have by 2.1

$$\begin{aligned} \tilde{b} &= (A - id)b^E + (A - id)b^\perp + r, \\ r &= (id - \tilde{B})b + A(id - B)A^{-1}a. \end{aligned}$$

If β is a translation, then $B = id = \tilde{B}$ and therefore $r = 0$.

If β is not a translation, then $|a| \leq |b|$ (by the choice of α) and therefore $|r| \leq (m(\tilde{B}) + m(B)) |b| \leq 2m(B) |b| < 4m(B) |b^E|$. Hence, in either case,

$$|r| < \frac{1}{2}(m(A) - m^\perp(A)) |b^E|.$$

Together with 2.2 (iv) we obtain

$$|\tilde{b}^\perp| < \frac{1}{2}(m(A) + m^\perp(A)) |b^E| < |\tilde{b}^E|.$$

We find that $\tilde{\beta}$ also satisfies (*), but with $|\tilde{b}| \leq m(A)|b| - r < |b|$, a contradiction.

3.3. *End of proof.* Lemma A provides elements in G with n linearly independent translation parts whose rotation parts are smaller than $\frac{1}{2}$. By Lemma B these elements are pure translations.

4. LATTICES

In this paragraph we collect the rudiments from lattice point theory which are necessary for the proof of Theorem II. A lattice L is a crystallographic group which consists only of translations. The elements of L (lattice points) will be identified with vectors in \mathbf{R}^n . By abuse of notation, we shall write $\omega = w = \text{trans } \omega$ for $\omega \in L$. It is well known that L is isomorphic to \mathbf{Z}^n but this fact will *not* be used in our proof of Theorem II. Notice, however that L is abelian and that the minimal distance of lattice points equals the length of the smallest non-zero element in L .

4.1. LEMMA. *Let L be a lattice in \mathbf{R}^n whose elements have pairwise distances ≥ 1 , and let $N(\rho)$ denote the number of lattice points in L whose distance from the origin is $\leq \rho$ ($\rho > 0$). Then*

$$N(\rho) \leq (2\rho + 1)^n.$$

Proof. The open balls of radius $\frac{1}{2}$ around the $N(\rho)$ lattice points are pairwise disjoint and all contained in a ball of radius $\rho + \frac{1}{2}$. Comparing the volumes we find $N(\rho) \left(\frac{1}{2}\right)^n \leq \left(\rho + \frac{1}{2}\right)^n$.