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use a method which is more in the spirit of Minkowski's geometry of numbers, from where Bieberbach's original arguments departed.

Since the material is standard, the exposition will be condensed. Yet some efforts have been made not to frustrate the reader by omitting details.

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2. Rigid motions

In this section we fix the notation and collect the necessary (and hopefully sufficient) rudiments from Linear Algebra.

We consider $\mathbf{R}^n$ as an euclidean vector space with the standard inner product. We use $|x|$ to denote the length of a vector $x \in \mathbf{R}^n$, and $\sphericalangle (x, y) \in [0, \pi]$ to denote the angle between two vectors. A rigid motion α (isometry of $\mathbf{R}^n$) will be expressed in the form

$$x \mapsto \alpha x = Ax + a \qquad (x \in \mathbf{R}^n)$$

where $A = \operatorname{rot} \alpha \in O(n)$ is an orthogonal map, called the *rotation part* of α, and $\alpha = \operatorname{trans} \alpha \in \mathbf{R}^n$ is a vector, called the *translation part.*

2.1. *The commutator* $[\alpha, \beta]$ of two rigid motions $x \mapsto \alpha x = Ax + a$ and $x \mapsto \beta x = Bx + b$ is defined as $[\alpha, \beta] = \alpha\beta\alpha^{-1}\beta^{-1}$. The following formulae are easily checked:

$$\operatorname{rot} [\alpha, \beta] = [A, B]$$
$$\operatorname{trans} [\alpha, \beta] = (A - id)b + (id - [A, B])b + A(id - B)A^{-1}a\,.$$

2.2. *Rotations.* For $A \in O(n)$ we define

$$m(A) = \max\left\{ |Ax - x| / |x| \;\middle|\; x \in \mathbf{R}^n \backslash \{0\} \right\}.$$

Note that $|Ax - x| \leqslant m(A)\,|x|$ for $x \in \mathbf{R}^n$. *The set*

(i) $$E_A = \left\{ x \in \mathbf{R}^n \;\middle|\; |Ax - x| = m(A)\,|x| \right\}$$

is a non trivial A-invariant subspace. This is immediately checked except perhaps for the part "$x, y \in E_A$ implies $x \pm y \in E_A$". This part follows from the equation

$$2m^2(A)\,(|x|^2 + |y|^2) = 2(|Ax - x|^2 + |Ay - y|^2) = |A(x+y) - (x+y)|^2$$
$$+ |A(x-y) - (x-y)|^2 \leqslant m^2(A)\,(|x+y|^2 + |x-y|^2) = 2m^2(A)\,(|x|^2 + |y|^2)$$

Since A is an orthogonal map, the orthogonal complement $E_A^\perp$ of E_A is also an A-invariant linear subspace of $\mathbf{R}^n$. We define

(ii) $$m^\perp(A) = \max \{ \mid Ax - x \mid / \mid x \mid \; \mid \; x \in E_A^\perp \setminus \{0\} \}$$

if $E_A^\perp \neq \{0\}$ and set $m^\perp(A) = 0$ if $E_A^\perp = \{0\}$. It follows that

(iii) $$m^\perp(A) < m(A) \text{ if } A \neq id \, .$$

We let $x = x^E + x^\perp$, $x^E \in E_A$, $x^\perp \in E_A^\perp$ be the orthogonal decomposition of a vector x with respect to E_A and $E_A^\perp$. The simple observation

(iv) $$\mid Ax^E - x^E \mid = m(A) \mid x^E \mid , \quad \mid Ax^\perp - x^\perp \mid \leqslant m^\perp(A) \mid x^\perp \mid$$

together with (iii), will play a crucial role in the proof of Theorem I.

2.3. *Commutator estimate. For* $A, B \in O(n)$ *we have*

$$m([A, B]) \leqslant 2m(A)\, m(B) \, .$$

Proof. Verify the identity

$$[A, B] - id = \big((A - id)(B - id) - (B - id)(A - id)\big) A^{-1} B^{-1}$$

From $\mid A^{-1} B^{-1} x \mid = \mid x \mid$ it then follows that

$$\mid [A, B]\, x - x \mid \leqslant m(A)\, m(B) \mid x \mid + m(B)\, m(A) \mid x \mid$$

for all $x \in \mathbf{R}^n$.

2.4. *Crystallographic groups.* Discreteness and compactness of the fundamental domain will be used as follows:

A group G of rigid motions in $\mathbf{R}^n$ is called crystallographic if

(i) for all $t > 0$ only finitely many $\alpha \in G$ have $\mid a \mid \leqslant t$,

(ii) there is some constant d such that for each $x \in \mathbf{R}^n$ there is an element $\alpha \in G$ satisfying $\mid a - x \mid \leqslant d$.

3. Proof of Theorem I

Now let G be an n-dimensional crystallographic group.

3.1. Lemma A ("Mini Bieberbach"). *For each unit vector* $u \in \mathbf{R}^n$ *and for all* $\varepsilon, \delta > 0$ *there exists* $\beta \in G$ *satisfying*