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SULLIVAN'S LAMINATION OF A PLANAR REGION

by Jan Mycielski

0. Introduction

Let R be a bounded open set on the plane $\mathbf{R}^2$, and B the boundary of R. For any $X \subset \mathbf{R}^2$, $H(X)$ denotes the convex hull of X. Our aim is to give a simple proof of the following Theorem.

THEOREM. *For every* $p \in R$ *there exists a circle* K *such that the open region bounded by* K *is included in* R *and* $p \in H(K \cap B)$.

This theorem yields a division of R into convex sets, which is called a lamination. D. Sullivan has proved [2] the existence of a similar lamination, where $H(K \cap B)$ is replaced by the Poincaré hull of $K \cap B$ relative to K, i.e., the open region bounded by K less all the open regions D bounded by circles orthogonal to K and such that $D \cap K \cap B = \emptyset$. (The proof of the existence of Sullivan's lamination is an easy modification of ours and will be left to the interested reader.)

Sullivan's proof is based upon certain properties of approximations of B by some curves in $\mathbf{R}^3$ which seem quite difficult. Our proof is based on the same idea but the implementation is much easier, by a natural application of nonstandard analysis. (I think that this proof may constitute a good introduction to that method since, the simplification which it brings is quite dramatic, and since I have tried to formulate it as closely as possible to the original idea of nonstandard analysis which was told to me by A. A. Robinson in 1962. Many subsequent presentations of nonstandard analysis seem more distant from that original idea and, in some sense, are less direct. The only prerequisites necessary for a full understanding of our proof are the notions of elementary extensions and ultrapowers, see e.g. [1].)

1. A GEOMETRIC LEMMA. Let F be a closed subset on the geometric sphere S^2. By the spherical convex hull of F, denoted $SH(F)$, we mean the intersection of all closed half-spheres of S^2 which include F. Let R be an

open subset of the southern hemisphere of S^2 and B the boundary of R. We have the following analog of the above Theorem.

PROPOSITION. *For every* $p \in R$ *there is a circle* $K \subseteq S^2$ *such that the open cap bounded by* K *is included in* R *and there are three points* $q_0, q_1, q_2 \in K \cap B$ *such that* $p \in SH\{q_0, q_1, q_2\}$.

Proof. Let $H(B)$ be the usual convex hull of B in $\mathbf{R}^3$. Then the radius connecting p to the center of S^2 intersects $H(B)$. Let q be a point of that intersection which is nearest to p. Thus q is on the boundary of $H(B)$. Let P be a plane supporting $H(B)$ at q. It is clear that the open half-space bounded by P and containing p does not intersect B and its intersection with S^2 is a spherical cap included in R. Let K be the circle $P \cap S^2$, i.e., the boundary of this cap. Of course there are $q_0, q_1, q_2 \in K \cap B$ such that $q \in H\{q_0, q_1, q_2\}$. Since a projection from the center of S^2 to S^2 turns convex hulls of subsets of the southern hemisphere into their spherical convex hulls, we have also $p \in SH\{q_0, q_1, q_2\}$.

Q.E.D.

Now we can quickly derive the Theorem from the Proposition using some standard methods of nonstandard analysis; but for convenience of the reader we add some preliminaries about those methods.

2. PRELIMINARIES. Let $\langle V, \in \rangle$ be Cantor's inverse, where $\in$ is the ordinary membership relation, and let $\langle W, \tilde{\in} \rangle$ be an elementary extension of $\langle V, \in \rangle$ such that $\mathbf{R}$ is a nonarchimedean field. {Since we will work within $\langle W, \tilde{\in} \rangle$ we chose to denote the standard membership relation by $\in$. Thus $x \in X$ implies $x \tilde{\in} X$ but not vice versa. E.g., $\mathbf{R}$ is an element of V, but there is no X in W such that $x \tilde{\in} X \leftrightarrow x \in \mathbf{R}$, unless both x and X are in V. In fact $\langle W, \tilde{\in} \rangle$ thinks that $\mathbf{R}$ is archimedean, just like in the "paradox" of Skolem an elementary submodel $\langle V', \in' \rangle$ of $\langle V, \in \rangle$ thinks that $\mathbf{R}$ is uncountable while $\{x : x \in' \mathbf{R}\}$ and V' are countable.} $\langle W, \tilde{\in} \rangle$ can be any proper ultrapower extension of $\langle V, \in \rangle$ with a countable exponent (see e.g., [1]). The elements of V are called the *standard* objects and the elements of $W \setminus V$ the *nonstandard* objects. The positive elements $x \tilde{\in} \mathbf{R}$ such that $x < y$ for all $y \in \mathbf{R}$ are called the (positive) *infinitesimals*. Their inverses are called *infinite reals*. It is easy to check that

(i) If $p \tilde{\in} \mathbf{R}^n$ and p is at a finite distance from the origin, then there exists a unique point $s(p) \in \mathbf{R}^n$ which is infinitesimally close to p.

Of course if $x \in X \leftrightarrow x \in Y$ and both X and Y are standard then $X = Y$. Hence, since V is closed under subsects,. for every $X \subseteq \mathbf{R}^n$ in W there exists a unique standard set $s[X]$ such that

$$y \in s[X] \leftrightarrow \exists x \,\tilde{\in}\, X \,[s(x) = y].$$

(ii) For every $X \subseteq \mathbf{R}^n$, $s[X]$ is a closed set.

Proof. We can assume that $X \neq \emptyset$. Since $s[X]$ is standard it is enough to show that for every standard $p \in \mathbf{R}^n$, $p \notin s[X]$, there exists a standard ball with center p disjoint with $s[X]$. Let U be the maximal ball in W with center p disjoint with X. The radius of U is not an infinitesimal nor 0 since otherwise we would have $p \in s[X]$, as there would be points of X whose distance from p is infinitesimal. Thus U includes a ball centered at p with some positive standard radius r. A ball centered at p with radius $r/2$ is still standard and disjoint with $s[X]$.

Q.E.D.

By related argument we can show

(iii) If $K \subseteq \mathbf{R}^n$, K is a circle, the center of K is at a finite distance from the origin and the radius of K is finite, but not infinitesimal, then $s[K]$ is also a circle.

(In these preliminaries I have followed the precept of Robinson:

"to take full advantage of my idea you should identify standard objects with their nonstandard extensions, and study their standard extensions from this wider point of view." Later he did not always follow this idea (at least in his notations), distinguishing e.g., the standard real line $\mathbf{R}$ from a non-standard one $\mathbf{R}^*$, i.e., he accepted an elementary mapping $*$: $\langle V, \in \rangle \to \langle W, \tilde{\in} \rangle$ instead of an elementary inclusion. I prefer to use the inclusion.)

3. Proof of the Theorem. Let R, B be as in the Theorem and $p \in R$. We take a sphere S^2 of infinite radius resting on $\mathbf{R}^2$ at $(0, 0)$. Let pr be the projection orthogonal to $\mathbf{R}^2$ of R and B into S^2. Of course $pr(R \bigcup B)$ is in the southern hemisphere of S^2 and is infinitely close to $R \bigcup B$. We apply the Proposition to $pr(R)$ and $pr(p)$. Thus we get a circle K' in S^2 such that one open cap C bounded by K' is in $pr(R)$, and there are three points $q_0, q_1, q_2 \,\tilde{\in}\, K' \bigcap pr(B)$ such that $pr(p) \,\tilde{\in}\, SH\{q_0, q_1, q_2\}$. Now it is easy to check that $s[K']$ is a standard circle in $\mathbf{R}^2$,

$$s[C] \subseteq s[pr(R)] = B \bigcup R\,,$$
$$s(q_0), s(q_1), s(q_2) \in s[K' \bigcap pr(B)] = s[K'] \bigcap B\,,$$

and

$$p = s(pr(p)) \in s[SH\{q_0, q_1, q_2\}] = H\{s(q_0), s(q_1), s(q_2)\}\,.$$

Q.E.D.

Remarks. 1. Instead of our infinite S^2 Sullivan used a sequence of growing spheres and had to prove that the circles K' in those spheres, given by the Proposition, converge to the desired circle K. This was more difficult; in fact he was dealing with the Poincaré analog of the above.

2. The Theorem generalizes to bounded open sets in $\mathbf{R}^n$ without any new ideas.

References

[1] Bell, J. L. and A. B. Slomson. *Models and Ultraproducts.* North-Holland Publ. Co., 1969.
[2] Sullivan, D. A lecture given at the University of Colorado in Boulder in 1981.

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