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**Autor:** Duddy, John  
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**THEOREM 4.** (i) Let  $K$  be a convolution operator of type  $r$  for  $r > 0$ . Then  $K$  extends from  $C_c^\infty(\mathbf{F}^m)$  to a bounded operator from  $L^p(\mathbf{F}^m)$  to  $L^q(\mathbf{F}^{m'})$  where  $1 < p < Q/r$  and  $q^{-1} = p^{-1} - r/Q$ . (ii) Let  $K$  be a convolution operator of type 0. Then  $K$  extends from  $C_c^\infty(\mathbf{F}^m)$  to a bounded operator from  $S_k^p(\mathbf{F}^m)$  to  $S_k^p(\mathbf{F}^{m'})$ .

Finally, we mention the interaction between the homogeneous convolution operators and the left-invariant differential operators. Let  $D: C^\infty(\mathbf{F}^{m'}) \rightarrow C^\infty(\mathbf{F}^{m''})$  be a left-invariant homogeneous differential operator of degree 1 and let  $K$  be a homogeneous convolution operator of type  $r$ , with  $r \geq 1$ . Then  $DK$  is a homogeneous convolution operator of type  $r - 1$ . Moreover, if  $r > 1$  the kernel of  $DK$  is given by  $Dk(x)$ .

#### 4. THE HODGE DECOMPOSITION

Consider the complex (1) where  $E_i = \mathbf{R}^n \times \mathbf{F}^{m_i}$ . Assume that each of the  $D_i$  is a first order, left-invariant operator, homogeneous of degree 1. So each entry of  $D_i$  is of the form  $\sum_{j=1}^d a_j X_{1,j}$  where  $a_j$  is constant. Construct the Laplacian,  $\Delta$ , with respect to the euclidian inner products on  $\mathbf{F}^{m_i}$ ,  $i = 1, 2, 3$ . Assume there exists a homogeneous convolution operator of type 2,  $K$ , which inverts  $\Delta$ . If  $f \in C_c^\infty(\mathbf{F}^{m_2})$  then  $f(x) = \Delta K f(x) = K \Delta f(x)$ .

**THEOREM 5.** Let  $f \in S_2^2(\mathbf{F}^{m_2})$ . As distributions,  $\Delta f = 0$  if and only if  $f = 0$ .

*Proof.* Obviously, if  $f = 0$  then  $\Delta f = 0$ .

Assume  $\Delta f = 0$ . Let  $\{f_j\}$  be a sequence in  $C_c^\infty(\mathbf{F}^{m_2})$  such that  $f_j \rightarrow f$  in  $S_2^2(\mathbf{F}^{m_2})$ . Then  $f_j \rightarrow f$  in the sense of distributions. Moreover,  $\Delta f_j \rightarrow \Delta f = 0$  in  $L^2(\mathbf{F}^{m_2})$ . Let  $g \in C_c^\infty(\mathbf{F}^{m_2})$ . Then

$$\langle f, g \rangle = \lim_{j \rightarrow \infty} \langle f_j, g \rangle = \lim_{j \rightarrow \infty} \langle f_j, \Delta K g \rangle = \lim_{j \rightarrow \infty} \langle \Delta f_j, K g \rangle.$$

Because  $g \in C_c^\infty(\mathbf{F}^{m_2})$  it is in  $L^p$  where  $p = 2Q/(Q+4)$ . Therefore, by Theorem 4(i),  $Kg \in L^q$  where

$$q^{-1} = (Q+4)/2Q - 2/Q = 1/2, \text{ i.e., } Kg \in L^2(\mathbf{F}^{m_2}).$$

For  $Q \geq 5$ ,  $1 < p < q < \infty$ . So

$$|\langle f, g \rangle| = \lim_{j \rightarrow \infty} |\langle \Delta f_j, K g \rangle| \leq \lim_{j \rightarrow \infty} \|\Delta f_j\|_{L^2(\mathbf{F}^{m_2})} \|Kg\|_{L^2(\mathbf{F}^{m_2})} = 0.$$

So, as a distribution,  $f = 0$ . This proves the theorem.

We have shown that the only harmonic element in  $S_2^2(\mathbf{F}^{m_2})$  is the zero element. Let  $f \in S_2^2(\mathbf{F}^{m_2})$  and let  $f_j \rightarrow f$  in  $S_2^2(\mathbf{F}^{m_2})$  with  $f_j \in C_c^\infty(\mathbf{F}^{m_2})$ . Then

$$f = \lim_{j \rightarrow \infty} \Delta K f_j = \lim_{j \rightarrow \infty} D_1 D_1^* K f_j + \lim_{j \rightarrow \infty} D_2^* D_2 K f_j = D_1 D_1^* K f + D_2^* D_2 K f .$$

To complete the Hodge decomposition we must prove that

$$D_1 D_1^* K f \perp D_2^* D_2 K f .$$

We need the following notation. Let  $D(R) = \{x \in N : |x| < R\}$  and  $S(R) = \{x \in N : |x| = R\}$ . Endow each set with the left-invariant metric induced by  $N$ . The metric gives rise to the corresponding volume elements which, in the case of  $D(R)$ , is the restriction of  $dx$ . Let  $d\mu_R$  denote the volume element on  $S(R)$ . For  $f, g \in C_c^\infty(D(R), \mathbf{F}^{m_i})$  define

$$(f, g)_{D(R), i} = \int_{D(R)} (f(x), g(x))_{i, x} dx$$

where  $(\cdot, \cdot)_{i, x}$  is the metric on  $\mathbf{F}^{m_i}$ . Similarly for  $f, g \in C^\infty(S(R), \mathbf{F}^{m_i})$  define

$$(f, g)_{S(R), i} = \int_{S(R)} (f(x), g(x))_{i, x} d\mu_R(x) .$$

By restriction, any element  $f \in C^\infty(N, \mathbf{F}^{m_i})$  gives rise to an element of  $C^\infty(S(R), \mathbf{F}^{m_i})$  or  $C^\infty(D(R), \mathbf{F}^{m_i})$ . In our notation, we will not distinguish  $f$  from its restrictions.

We will be integrating by parts on  $D(R)$  which will involve a boundary integral on  $S(R)$ . To that end we define the symbol of our differential operators. Define  $\xi(x) = |x| - R$  and let  $g \in C^1(N, \mathbf{F}^{m_i})$ . Let  $x \in S(R)$ . Then  $\xi(x) = 0$ . The symbol of  $D_i$  at  $x \in S(R)$  acting on  $d\xi$  and on  $g$  is given by

$$\sigma(D_i, d\xi)g(x) = D_i(\xi g)(x) .$$

The integration by parts formula is

$$(13) \quad (D_i g, f)_{D(R), i+1} = (g, D_i^* f)_{D(R), i} + (\sigma(D_i, d\xi)g, f)_{S(R), i} .$$

**THEOREM 6.** Assume  $f \in L^2(\mathbf{F}^{m_2}) \cap L^q(\mathbf{F}^{m_2})$  where  $q = Q/Q + 2$ . Then  $f = D_1 D_1^* K f + D_2^* D_2 K f$  and  $D_1 D_1^* K f \perp D_2^* D_2 K f$ .

*Proof.* We have already seen that  $f = D_1 D_1^* K f + D_2^* D_2 K f$ . To prove the orthogonality we restrict our attention to  $D(R)$  for  $R$  large.

For brevity let  $h = (D_1 D_1^* Kf, D_2^* D_2 Kf)_2$ . Also, let

$$h(R) = (D_1 D_1^* Kf, D_2^* D_2 Kf)_{D(R), 2}.$$

Then  $\lim_{R \rightarrow \infty} h(R) = h$ . Note that  $D_1 D_1^* K$  and  $D_2^* D_2 K$  are type 0 operators.

Since  $f \in L^2(\mathbf{F}^{m_2})$  by theorem 4 (ii) we know that  $h$  is defined. Furthermore  $h(R)$  is bounded for all  $R$  by  $\|D_1 D_1^* Kf\|_{L^2(\mathbf{F}^{m_2})} \|D_2^* D_2 Kf\|_{L^2(\mathbf{F}^{m_2})}$ .

We can compute  $h(R)$  as follows:

$$\begin{aligned} h(R) &= (D_1 D_1^* Kf, D_2^* D_2(Kf))_{D(R), 2} = (D_2 D_1 D_1^* Kf, D_2 Kf)_{D(R), 3} \\ &\quad + (D_1 D_1^* Kf, \sigma(D_2^*, d(|x|)) D_2 Kf)_{S(R), 2} \end{aligned}$$

by (13). We now prove a sequence of lemmas.

LEMMA 1.  $h(R)$  is continuous.

*Proof.* This follows from Lebesgue's dominated convergence theorem.

LEMMA 2. Let  $R \geq 1$ ,  $x \in N$  and  $|x| = R$ . Then

$$|\sigma(D_2^*, d|x|)g(x)| \leq C |g(x)|$$

where  $C$  is a constant independent of  $g$ .

*Proof.* Recall that  $X_1, \dots, X_d$  is our orthonormal basis for  $\mathfrak{n}_1$  where  $d = \dim(\mathfrak{n}_1)$ . The entries of  $D_2^*$  are linear combinations of the  $X_i$ ,  $i = 1, \dots, d$  with coefficients in  $\mathbf{F}$ . Let  $D_{ij}$  be the  $i, j$  entry,  $1 \leq i \leq m_2$ ,

$1 \leq j \leq m_3$ . Then  $D_{ij} = \sum_{k=1}^d C_{ij}^k X_k$ . Thus, for  $x \in S(R)$

$$\begin{aligned} |\sigma(D_2^*, d|x|)g(x)| &\leq C \sum_{i=1}^{m_2} \left| \sum_{j=1}^{m_3} D_{ij} (|x| - R) g_j(x) \right| \\ &\leq C \sum_{i,j,k} |C_{ij}^k X_k (|x| - R) g_j(x)| \\ &\leq C \sum_{jk} |(X_k |x|) g_j(x)| \quad (\text{since } |x| = R) \\ &\leq C (\max_k (X_k |x|)) |g_j(x)|. \end{aligned}$$

We must show  $X_k |x|$  is bounded. But  $|x|$  is  $C^\infty$  away from the origin and homogeneous of degree 1. So  $X_k |x|$  is homogeneous of degree 0. Thus, it is determined by its values on  $\{|x| = 1\}$ . It is  $C^\infty$  on this set and, therefore, bounded. This proves the lemma.

Since  $h(R) \rightarrow h$  as  $R \rightarrow \infty$  we have

LEMMA 3. For  $\varepsilon > 0$ ,  $\lim_{r \rightarrow \infty} \frac{1}{2\varepsilon} \int_{r-\varepsilon}^{r+\varepsilon} h(R)dR = h$ .

We continue with the proof of our theorem. By the preceding lemma it suffices for us to prove that  $\lim_{r \rightarrow \infty} \frac{1}{2\varepsilon} \int_{r-\varepsilon}^{r+\varepsilon} h(R)dR = 0$ . But,

$$(14) \quad \begin{aligned} & \frac{1}{2\varepsilon} \int_{r-\varepsilon}^{r+\varepsilon} h(R)dR \\ &= \frac{1}{2\varepsilon} \int_{r-\varepsilon \leq |x| \leq r+\varepsilon} (D_1 D_1^* Kf, \sigma(D_2^*, d|x|) D_2 Kf)_x \|d|x|\| dx \end{aligned}$$

because  $d\mu_R dR = \|d|x|\| dx$ . We claim that  $\|d|x|\|$  is bounded. Let  $\omega^{ij}$  be dual to  $X_{ij}$  where  $X_{ij}$  is our orthonormal basis. Then  $d|x| = \sum_{i,j} (X_{ij}|x|)\omega^{ij}$ . Since  $|x|$  is homogeneous of degree 1 and  $X_{ij}$  is homogeneous of degree  $i$  we have  $X_{ij}|x|$  is homogeneous of degree  $1 - i$ . Hence, for  $|x| \geq 1$  each  $X_{ij}|x|$  is bounded. So  $\|d|x|\|$  is bounded.

By assumption,  $f \in L^q(\mathbb{F}^{m_2})$ ,  $q = Q/Q + 2$  and we know that  $D_2 K$  is type 1. By Theorem 4(i) we know that  $D_2 Kf \in L^2(\mathbb{F}^{m_2})$ . Thus, by Lemma 2  $|\chi_r \sigma(D_2^*, d|x|) D_2 Kf| \leq C |\chi_r D_2 Kf|$  where  $\chi_r$  is the characteristic function of  $\{r - \varepsilon \leq |x| \leq r + \varepsilon\}$ . We conclude that  $\chi_r \sigma(D_2^*, d|x|) D_2 Kf \in L^2(\mathbb{F}^{m_2})$ . By the Schwarz inequality and the fact that  $\|d|x|\|$  is bounded, we get, from (14)

$$\frac{1}{2\varepsilon} \int_{r-\varepsilon}^{r+\varepsilon} h(R)dR \leq \frac{c}{2\varepsilon} \| \chi_r D_1 D_1^* Kf \|_{L^2(\mathbb{F}^{m_2})} \| \chi_r D_2 Kf \|_{L^2(\mathbb{F}^{m_3})} .$$

As  $r \rightarrow \infty$ , both  $\| \chi_r D_1 D_1^* Kf \|_{L^2(\mathbb{F}^{m_2})}$  and  $\| \chi_r D_2 Kf \|_{L^2(\mathbb{F}^{m_3})}$  tend to 0. So

$$h = \lim_{r \rightarrow \infty} \frac{1}{2\varepsilon} \int_{r-\varepsilon}^{r+\varepsilon} h(R)dR = 0. \text{ This proves the theorem.}$$

This theorem together with Theorem 5 proves the Hodge decomposition. A similar argument gives the solution to the problem of finding  $g$  such that  $D_1 g = f$  for a given  $f$ .

THEOREM 7. Let  $f \in L^2(\mathbb{F}^{m_2}) \cap L^q(\mathbb{F}^{m_2})$  with  $q = Q/Q + 2$ . Suppose  $D_2 f = 0$ . Then there exists  $g \in L^2(\mathbb{F}^{m_1})$  such that  $D_1 g = f$ .

Proof. We have  $f = D_1 D_1^* Kf + D_2^* D_2 Kf$ . It suffices to prove  $(f, D_2^* D_2 Kf) = 0$  because this implies  $D_2^* D_2 Kf = 0$  since

$$D_1 D_1^* Kf \perp D_2^* D_2 Kf .$$

We may set  $g = D_1^*Kf$ . Using the same notation as in the preceding theorem we have

$$\begin{aligned} (f, D_2^*D_2Kf)_2 &= \lim_{R \rightarrow \infty} (f, D_2^*D_2Kf)_{D(R), 2} \\ &= \lim_{R \rightarrow \infty} \left( (D_2f, D_2Kf)_{D(R), 3} + (f, \sigma(D_2^*, d|x|)D_2Kf)_{S(R), 2} \right) \\ &= \lim_{R \rightarrow \infty} (f, \sigma(D_2^*, d|x|)D_2Kf)_{S(R), 2} \quad (\text{since } D_2f = 0). \end{aligned}$$

The same argument as in Theorem 6 proves that the limit is zero. This proves the theorem.

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