Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 31 (1985)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: HODGE DECOMPOSITION ON STRATIFIED LIE GROUPS

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Kapitel: 3. Differential complexes on stratified groups

DOI: https://doi.org/10.5169/seals-54573

$$\text{(8)}\quad \text{ii)}\quad \bar\partial_b^*\phi = -\sum_{|J|=q}\sum_{j=1}^{n} Z_j\phi_J\bar\omega^j \lrcorner\, \bar\omega^J ,$$

$$\text{iii)}\quad \Box_b\phi = \sum_{|J|=q}\left(-\frac{1}{2}\sum_{j=1}^{n}(Z_j\bar Z_j+\bar Z_j Z_j) + i(n-2q)T\right)\phi_J\bar\omega^J .$$

Define the function

$$\Phi_\alpha(z,t) = (|z|^2-it)^{-\frac{n+\alpha}{2}}(|z|^2+it)^{-\frac{n-\alpha}{2}} .$$

Let $\phi\in\Lambda_c^{0,q}$, $q\neq 0, n$. For an appropriate constant, c_q, define

$$K_q\phi(v) = c_q\sum_{|J|=q}\left(\int_H \phi_J(u)\Phi_{n-2q}(u^{-1}v)du\right)\bar\omega^J . \tag{9}$$

Folland and Stein prove that for the appropriate c_q

THEOREM 1. *Let* $\phi\in\Lambda_c^{0,q}$, $q\neq 0, n$. *Then* $\Box_b K_q\phi = K_q\Box_b\phi = \phi$.

In [4] we prove a stronger version of the following Hodge decomposition theorem.

THEOREM 2. *Let* $\phi\in\Lambda_c^{0,q}$, $q\neq 0, n$. *Then*

i) $H\phi = 0$ *where* H *is the orthogonal projection onto the kernel of* $\Box_b$.

ii) $\phi = \bar\partial_b\bar\partial_b^* K_q\phi + \bar\partial_b^*\bar\partial_b K_q\phi$.

We also prove

THEOREM 3. *If* $\phi\in\Lambda_c^{0,q}$, $q\neq 0, n$ *and if* $\bar\partial_b\phi = 0$ *then* $\psi = \bar\partial_b^* K_q\phi$ *satisfies* $\bar\partial_b\psi = \phi$.

These two theorems are special cases of theorems 6 and 7 proven in section 4.

3. DIFFERENTIAL COMPLEXES ON STRATIFIED GROUPS

We study a class of nilpotent Lie groups which we describe in terms of their Lie algebras. A graded Lie algebra, $\mathfrak{n}$, is a finite dimensional nilpotent algebra which has a direct sum decomposition, $\mathfrak{n} = \bigoplus_{i=1}^{r}\mathfrak{n}_i$ where the $\mathfrak{n}_i$ satisfy

i) $[\mathfrak{n}_i,\mathfrak{n}_j]\subseteq\mathfrak{n}_{i+j}$ if $i+j\leqslant r$,

ii) $[\mathfrak{n}_i,\mathfrak{n}_j] = 0$ if $i+j > r$.

Let $n = \dim \mathfrak{n}$. Define the homogeneous dimension to be $Q = \sum_{j=1}^{r} j \dim(\mathfrak{n}_j)$. If $\mathfrak{n}$ is a graded algebra and if $\mathfrak{n}_1$ generates $\mathfrak{n}$ then $\mathfrak{n}$ is called a stratified algebra. A Lie group is called a stratified group if its Lie algebra is a stratified algebra. For a given stratified algebra $\mathfrak{n}$ we will restrict our attention to the simply connected group associated to it.

The Heisenberg group is a simply connected stratified group. In fact, identifying the Lie algebra with the left invariant vector fields, we may take $\mathfrak{n}_1$ to be the span of the X's and Y's and $\mathfrak{n}_2$ to be the span of T. By (3) and (4) we see that $[\mathfrak{n}_1, \mathfrak{n}_1] = \mathfrak{n}_2$ and $[\mathfrak{n}_1, \mathfrak{n}_2] = [\mathfrak{n}_2, \mathfrak{n}_2] = 0$.

Any graded nilpotent group has a natural family of dilations. First we define them on the Lie algebra. Let $X \in \mathfrak{n}$. Then by definition $X = \sum_{j=1}^{r} X_j$ where $X_j \in \mathfrak{n}_j$. For $s > 0$ set $\delta_s(X) = \sum_{j=1}^{r} s^j X_j$. Because $\mathfrak{n}$ is nilpotent the exponential map is globally defined. Suppose $x \in N$ and $x = \exp(X)$ for $X \in \mathfrak{n}$. Define $\delta_s(x) = \exp(\delta_s X)$. Suppose we are given an inner product on $\mathfrak{n}$ such that $\mathfrak{n}_i \perp \mathfrak{n}_j$ for all $i \neq j$. Let $\| X \|$ be the length defined by the inner product. Suppose $x = \exp(X)$ where $X = \sum_{j=1}^{r} X_j$, $X_j \in \mathfrak{n}_j$. Then define the homogeneous norm function to be

$$| x | = \left(\sum_{j=1}^{r} \| X_j \|^{\frac{2r!}{j}} \right)^{\frac{1}{2r!}}.$$

Then (i) $| x | = 0$ if and only if $x = 0$, (ii) $x \to | x |$ is continuous on N and C^∞ on $N - \{0\}$, (iii) $| \delta_s x | = s | x |$.

On the Heisenberg group, $\delta_s((z, t)) = (sz, s^2 t)$ and $| z | = (|z|^4 + t^2)^{\frac{1}{4}}$.

Recall that the homogeneous dimension is $Q = \sum_{j=1}^{r} j \dim(\mathfrak{n}_j)$. Let f be a function on N. We say f is homogeneous of degree p if $f(\delta_s(x)) = s^p f(x)$. If $-Q < p$ then such an f is in L^k_{loc} for $1 \leqslant k < \infty$. A distribution F is called homogeneous of degree p if

$$<F, s^{-Q} g(\delta_{s^{-1}} x)> = s^p <F, g>$$

where $g \in C_c^\infty(N)$ and $<F, g>$ is the pairing of $C_c^\infty(N)$ with its dual, $D'(N)$. A differential operator L (acting on functions) is homogeneous of degree p if $L(f \cdot \delta_s) = s^p (Lf) \circ \delta_s$. Observe that if f is a homogeneous function of degree p and if L is a homogeneous differential operator of degree p' then Lf is a homogeneous function of degree $p - p'$.

Let $X_{i,1}, \ldots, X_{i,\dim(\mathfrak{n}_i)}$ be an orthonormal basis of $\mathfrak{n}_i$ with respect to our inner product. Since $\mathfrak{n}_i \perp \mathfrak{n}_j$ for $i \neq j$ the set

$$\{X_{i,j}: 1 \leqslant i \leqslant r, 1 \leqslant j \leqslant \dim(\mathfrak{n}_i)\}$$

is an orthonormal basis of $\mathfrak{n}$. Define the global coordinate chart on N by

$$(x_{ij}) \to \Sigma x_{ij} X_{ij} \to \exp(\Sigma x_{ij} X_{ij}). \tag{10}$$

This identifies N with $\mathbf{R}^n$ as a manifold.

Let m_1, m_2 and m_3 be positive integers. For $i = 1, 2, 3$ define $E_i = \mathbf{R}^n \times \mathbf{F}^{m_i}$ to be the trivial bundle over $N = \mathbf{R}^n$ with fiber $\mathbf{F}^{m_i}$. Consider the differential complex (1). We know that each D_i can be expressed as an $m_{i+1} \times m_i$ matrix of differential operators on functions, $i = 1, 2$. If each entry is homogeneous of degree p we say D_i is a homogeneous differential operator of degree p. If each entry is left-invariant we say D_i is a left-invariant differential operator.

On our prototype, the Heisenberg group, we have the left-invariant metric which makes the Z's, $\bar{Z}$'s, and T into an orthonormal basis. Let $\bar{\omega}_1, \ldots, \bar{\omega}_n$ be a basis for $T^{0,1}$ which is dual to $\bar{Z}_1, \ldots, \bar{Z}_n$. Then

$$\{\bar{\omega}^J: J = (j_1, \ldots, j_q), 1 \leqslant j_1 < j_2 < \ldots < j_q \leqslant n\}$$

is a global orthonormal basis of $\Lambda^{0,q}$ for each q. So $\Lambda^{0,q}$ is a trivial bundle over $H \approx \mathbf{R}^{2n+1}$, and we may identify sections of $\Lambda^{0,q}$ with $C^\infty(\mathbf{R}^{2n+1}, \mathbf{C}^m)$ where $m = n!/q!(n-q)!$. By (8(iii)) the operator $\square_b: \Lambda^{0,q} \to \Lambda^{0,q}$ is given by the matrix $(\delta_{ij} L)_{1 \leqslant i,j \leqslant m}$ where $L = -\frac{1}{2} \sum_{k=1}^{n} (Z_j \bar{Z}_j + \bar{Z}_j Z_j) + i(n-2q)T$. L is left-invariant and homogeneous of degree 2. So, $\square_b$ is left-invariant and homogeneous of degree 2. Similarly, $K_q\phi$ defined by (9) can be written as

$$K_q\phi = \int_H c_q\, \Phi_{n-2q}(u^{-1}v) I \phi\, du$$

where $\phi \in \Lambda_c^{0,q}$ is a $q \times 1$ column vector and I is the $q \times q$ identity matrix. Note that Φ_{n-2q} is a homogeneous function of degree $-2n$. This example motivates the following definition of a homogeneous convolution operator.

Return to N, our stratified Lie group with global coordinates defined by (10). Let $k: N \to \mathrm{Mat}(m' \times m, \mathbf{F})$ be a mapping of N into the space of $m' \times m$ matrices with entries in $\mathbf{F}$. Given $f \in C_c^\infty(\mathbf{F}^m)$ and $x, y \in N$ the product $k(y^{-1}x) f(y)$ is an $m' \times 1$ column vector. We set

$$Kf(x) = \int_N k(y^{-1}x)f(y)dy\,. \tag{11}$$

The measure, dy, is the Haar measure on N. Under suitable restrictions on k the integral exists. The operator K is called a convolution operator with kernel k. If each entry of k is smooth away from 0 and homogeneous of degree $-Q + p$, $0 < p < Q$, we say that K is a homogeneous convolution operator of type p. As we mentioned before, a homogeneous function is in L^p_{loc} so the integral in (11) exists for $f \in C_c^\infty(\mathbf{F}^m)$.

Suppose k is homogeneous of degree $-Q$ and for each entry

$$k_{ij},\ 1 \leqslant i \leqslant m',\ 1 \leqslant j \leqslant m\,,$$

we have

$$\int_{a \leqslant |x| \leqslant b} k_{ij}(x)dx = 0 \tag{12}$$

for all a and b. We say an operator K is of type 0 if for some constant c we have

$$Kf(x) = \lim_{\varepsilon \to 0} \int_{\varepsilon \leqslant |y| \leqslant 1/\varepsilon} k(y^{-1}x)f(y)dy + cf(0) \quad \text{for all} \quad f \in C_c^\infty(\mathbf{F}^m)$$

where k satisfies (12). We refer the reader to Folland [9] or Rothschild and Stein [16] for details.

To study the continuity properties of these operators we define L^p spaces and Sobolev-type spaces of sections from N to $\mathbf{F}^m$. Let $\|\ \|_{L^p}$ denote the usual L^p norm on functions. Let $f \in C_c^\infty(\mathbf{F}^m)$ and let f_i, $i = 1, \dots, m$ be the components of f. Define the norm

$$\| f \|_{L^p(F^m)} = \left(\sum_{i=1}^m \| f_i \|_{L^p}^p \right)^{1/p}.$$

Let $L^p(\mathbf{F}^m)$ be the completion of $C_c^\infty(\mathbf{F}^m)$ under this norm.

Let $\{X_{1,1}, \dots, X_{1,d}\}$ be the orthonormal basis of $\mathfrak{n}_1$, with $d = \dim(\mathfrak{n}_1)$. For brevity, we will drop reference to the first subscript. Let J be a multi-index, $J = (j_1, j_2, \dots, j_q)$ with $1 \leqslant j_1 < j_2 < \dots < j_q \leqslant d$. Define $|J| = q$ and define $X_J = X_{j_1}X_{j_2} \dots X_{j_q}$. Define $S_q^p(\mathbf{F}^m)$ to be the closure of $C_c^\infty(\mathbf{F}^m)$ under the norm

$$\| f \|_{S_q^p(F^m)} = \left(\| f \|_{L^p(F^m)}^p + \sum_{i=1}^m \sum_{|J| \leqslant q} \| X_J f_i \|_{L^p}^p \right)^{1/p}.$$

A modification of a theorem by Folland [9] yields

THEOREM 4. (i) *Let* K *be a convolution operator of type* r *for* $r > 0$. *Then* K *extends from* $C_c^\infty(\mathbf{F}^m)$ *to a bounded operator from* $L^p(\mathbf{F}^m)$ *to* $L^q(\mathbf{F}^{m'})$ *where* $1 < p < Q/r$ *and* $q^{-1} = p^{-1} - r/Q$. (ii) *Let* K *be a convolution operator of type* 0. *Then* K *extends from* $C_c^\infty(\mathbf{F}^m)$ *to a bounded operator from* $S_k^p(\mathbf{F}^m)$ *to* $S_k^p(\mathbf{F}^{m'})$.

Finally, we mention the interaction between the homogeneous convolution operators and the left-invariant differential operators. Let $D: C^\infty(\mathbf{F}^{m'}) \to C^\infty(\mathbf{F}^{m''})$ be a left-invariant homogeneous differential operator of degree 1 and let K be a homogeneous convolution operator of type r, with $r \geqslant 1$. Then DK is a homogeneous convolution operator of type $r - 1$. Moreover, if $r > 1$ the kernel of DK is given by $Dk(x)$.

4. THE HODGE DECOMPOSITION

Consider the complex (1) where $E_i = \mathbf{R}^n \times \mathbf{F}^{m_i}$. Assume that each of the D_i is a first order, left-invariant operator, homogeneous of degree 1. So each entry of D_i is of the form $\sum_{j=1}^{d} a_j X_{1,j}$ where a_j is constant. Construct the Laplacian, Δ, with respect to the euclidian inner products on $\mathbf{F}^{m_i}$, $i = 1, 2, 3$. Assume there exists a homogeneous convolution operator of type 2, K, which inverts Δ. If $f \in C_c^\infty(\mathbf{F}^{m_2})$ then $f(x) = \Delta K f(x) = K \Delta f(x)$.

THEOREM 5. *Let* $f \in S_2^2(\mathbf{F}^{m_2})$. *As distributions,* $\Delta f = 0$ *if and only if* $f = 0$.

Proof. Obviously, if $f = 0$ then $\Delta f = 0$.

Assume $\Delta f = 0$. Let $\{f_j\}$ be a sequence in $C_c^\infty(\mathbf{F}^{m_2})$ such that $f_j \to f$ in $S_2^2(\mathbf{F}^{m_2})$. Then $f_j \to f$ in the sense of distributions. Moreover, $\Delta f_j \to \Delta f = 0$ in $L^2(\mathbf{F}^{m_2})$. Let $g \in C_c^\infty(\mathbf{F}^{m_2})$. Then

$$<f, g> = \lim_{j \to \infty} <f_j, g> = \lim_{j \to \infty} <f_j, \Delta K g> = \lim_{j \to \infty} <\Delta f_j, K g> .$$

Because $g \in C_c^\infty(\mathbf{F}^{m_2})$ it is in L^p where $p = 2Q/(Q+4)$. Therefore, by Theorem 4(i), $Kg \in L^q$ where

$$q^{-1} = (Q+4)/2Q - 2/Q = 1/2, \text{ i.e., } Kg \in L^2(\mathbf{F}^{m_2}) .$$

For $Q \geqslant 5$, $1 < p < q < \infty$. So

$$| <f, g> | = \lim_{j \to \infty} | <\Delta f_j, Kg> | \leqslant \lim_{j \to \infty} \| \Delta f_j \|_{L^2(\mathbf{F}^{m_2})} \| Kg \|_{L^2(\mathbf{F}^{m_2})} = 0 .$$