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Autor: Duddy, John
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$$(8) \quad \text{ii) } \bar{\partial}_b^* \phi = - \sum_{|J|=q} \sum_{j=1}^n Z_j \phi_J \bar{\omega}^j \lrcorner \bar{\omega}^J,$$

$$\text{iii) } \square_b \phi = \sum_{|J|=q} \left(-\frac{1}{2} \sum_{j=1}^n (Z_j \bar{Z}_j + \bar{Z}_j Z_j) + i(n-2q)T \right) \phi_J \bar{\omega}^J.$$

Define the function

$$\Phi_\alpha(z, t) = (|z|^2 - it)^{-\frac{n+\alpha}{2}} (|z|^2 + it)^{-\frac{n-\alpha}{2}}.$$

Let $\phi \in \Lambda_c^{0,q}$, $q \neq 0$, n . For an appropriate constant, c_q , define

$$(9) \quad K_q \phi(v) = c_q \sum_{|J|=q} \left(\int_H \phi_J(u) \Phi_{n-2q}(u^{-1}v) du \right) \bar{\omega}^J.$$

Folland and Stein prove that for the appropriate c_q

THEOREM 1. *Let $\phi \in \Lambda_c^{0,q}$, $q \neq 0$, n . Then $\square_b K_q \phi = K_q \square_b \phi = \phi$.*

In [4] we prove a stronger version of the following Hodge decomposition theorem.

THEOREM 2. *Let $\phi \in \Lambda_c^{0,q}$, $q \neq 0$, n . Then*

- i) $H\phi = 0$ where H is the orthogonal projection onto the kernel of \square_b .
- ii) $\phi = \bar{\partial}_b \bar{\partial}_b^* K_q \phi + \bar{\partial}_b^* \bar{\partial}_b K_q \phi$.

We also prove

THEOREM 3. *If $\phi \in \Lambda_c^{0,q}$, $q \neq 0$, n and if $\bar{\partial}_b \phi = 0$ then $\psi = \bar{\partial}_b^* K_q \phi$ satisfies $\bar{\partial}_b \psi = \phi$.*

These two theorems are special cases of theorems 6 and 7 proven in section 4.

3. DIFFERENTIAL COMPLEXES ON STRATIFIED GROUPS

We study a class of nilpotent Lie groups which we describe in terms of their Lie algebras. A graded Lie algebra, \mathfrak{n} , is a finite dimensional nilpotent algebra which has a direct sum decomposition, $\mathfrak{n} = \bigoplus_{i=1}^r \mathfrak{n}_i$ where the \mathfrak{n}_i satisfy

- i) $[\mathfrak{n}_i, \mathfrak{n}_j] \subseteq \mathfrak{n}_{i+j}$ if $i + j \leq r$,
- ii) $[\mathfrak{n}_i, \mathfrak{n}_j] = 0$ if $i + j > r$.

Let $n = \dim n$. Define the homogeneous dimension to be $Q = \sum_{j=1}^r j \dim(n_j)$.

If n is a graded algebra and if n_1 generates n then n is called a stratified algebra. A Lie group is called a stratified group if its Lie algebra is a stratified algebra. For a given stratified algebra n we will restrict our attention to the simply connected group associated to it.

The Heisenberg group is a simply connected stratified group. In fact, identifying the Lie algebra with the left invariant vector fields, we may take n_1 to be the span of the X 's and Y 's and n_2 to be the span of T . By (3) and (4) we see that $[n_1, n_1] = n_2$ and $[n_1, n_2] = [n_2, n_2] = 0$.

Any graded nilpotent group has a natural family of dilations. First we define them on the Lie algebra. Let $X \in n$. Then by definition $X = \sum_{j=1}^r X_j$

where $X_j \in n_j$. For $s > 0$ set $\delta_s(X) = \sum_{j=1}^r s^j X_j$. Because n is nilpotent the exponential map is globally defined. Suppose $x \in N$ and $x = \exp(X)$ for $X \in n$. Define $\delta_s(x) = \exp(\delta_s X)$. Suppose we are given an inner product on n such that $n_i \perp n_j$ for all $i \neq j$. Let $\|X\|$ be the length defined by the inner product. Suppose $x = \exp(X)$ where $X = \sum_{j=1}^r X_j$, $X_j \in n_j$. Then define the homogeneous norm function to be

$$|x| = \left(\sum_{j=1}^r \|X_j\|^{\frac{2r!}{j}} \right)^{\frac{1}{2r!}}$$

Then (i) $|x| = 0$ if and only if $x = 0$, (ii) $x \rightarrow |x|$ is continuous on N and C^∞ on $N - \{0\}$, (iii) $|\delta_s x| = s|x|$.

On the Heisenberg group, $\delta_s((z, t)) = (sz, s^2t)$ and $|z| = (|z|^4 + t^2)^{\frac{1}{4}}$.

Recall that the homogeneous dimension is $Q = \sum_{j=1}^r j \dim(n_j)$. Let f be a function on N . We say f is homogeneous of degree p if $f(\delta_s(x)) = s^p f(x)$. If $-Q < p$ then such an f is in L^k_{loc} for $1 \leq k < \infty$. A distribution F is called homogeneous of degree p if

$$\langle F, s^{-Q} g(\delta_{s^{-1}} x) \rangle = s^p \langle F, g \rangle$$

where $g \in C_c^\infty(N)$ and $\langle F, g \rangle$ is the pairing of $C_c^\infty(N)$ with its dual, $D'(N)$. A differential operator L (acting on functions) is homogeneous of degree p if $L(f \cdot \delta_s) = s^p(Lf) \circ \delta_s$. Observe that if f is a homogeneous function of degree p and if L is a homogeneous differential operator of degree p' then Lf is a homogeneous function of degree $p - p'$.

Let $X_{i,1}, \dots, X_{i, \dim(n_i)}$ be an orthonormal basis of n_i with respect to our inner product. Since $n_i \perp n_j$ for $i \neq j$ the set

$$\{X_{i,j}: 1 \leq i \leq r, 1 \leq j \leq \dim(n_i)\}$$

is an orthonormal basis of n . Define the global coordinate chart on N by

$$(10) \quad (x_{ij}) \rightarrow \sum x_{ij} X_{ij} \rightarrow \exp(\sum x_{ij} X_{ij}).$$

This identifies N with \mathbf{R}^n as a manifold.

Let m_1, m_2 and m_3 be positive integers. For $i = 1, 2, 3$ define $E_i = \mathbf{R}^n \times \mathbf{F}^{m_i}$ to be the trivial bundle over $N = \mathbf{R}^n$ with fiber \mathbf{F}^{m_i} . Consider the differential complex (1). We know that each D_i can be expressed as an $m_{i+1} \times m_i$ matrix of differential operators on functions, $i = 1, 2$. If each entry is homogeneous of degree p we say D_i is a homogeneous differential operator of degree p . If each entry is left-invariant we say D_i is a left-invariant differential operator.

On our prototype, the Heisenberg group, we have the left-invariant metric which makes the Z 's, \bar{Z} 's, and T into an orthonormal basis. Let $\bar{\omega}_1, \dots, \bar{\omega}_n$ be a basis for $T^{0,1}$ which is dual to $\bar{Z}_1, \dots, \bar{Z}_n$. Then

$$\{\bar{\omega}^J: J = (j_1, \dots, j_q), 1 \leq j_1 < j_2 < \dots < j_q \leq n\}$$

is a global orthonormal basis of $\Lambda^{0,q}$ for each q . So $\Lambda^{0,q}$ is a trivial bundle over $H \approx \mathbf{R}^{2n+1}$, and we may identify sections of $\Lambda^{0,q}$ with $C^\infty(\mathbf{R}^{2n+1}, \mathbf{C}^m)$ where $m = n!/q!(n-q)!$. By (8(iii)) the operator $\square_b: \Lambda^{0,q} \rightarrow \Lambda^{0,q}$ is given by the matrix $(\delta_{ij} L)_{1 \leq i, j \leq m}$ where $L = -\frac{1}{2} \sum_{k=1}^n (Z_k \bar{Z}_k + \bar{Z}_k Z_k) + i(n-2q)T$. L is left-invariant and homogeneous of degree 2. So, \square_b is left-invariant and homogeneous of degree 2. Similarly, $K_q \phi$ defined by (9) can be written as

$$K_q \phi = \int_H c_q \Phi_{n-2q}(u^{-1}v) I \phi du$$

where $\phi \in \Lambda_c^{0,q}$ is a $q \times 1$ column vector and I is the $q \times q$ identity matrix. Note that Φ_{n-2q} is a homogeneous function of degree $-2n$. This example motivates the following definition of a homogeneous convolution operator.

Return to N , our stratified Lie group with global coordinates defined by (10). Let $k: N \rightarrow \text{Mat}(m' \times m, \mathbf{F})$ be a mapping of N into the space of $m' \times m$ matrices with entries in \mathbf{F} . Given $f \in C_c^\infty(\mathbf{F}^m)$ and $x, y \in N$ the product $k(y^{-1}x)f(y)$ is an $m' \times 1$ column vector. We set

$$(11) \quad Kf(x) = \int_N k(y^{-1}x)f(y)dy .$$

The measure, dy , is the Haar measure on N . Under suitable restrictions on k the integral exists. The operator K is called a convolution operator with kernel k . If each entry of k is smooth away from 0 and homogeneous of degree $-Q + p$, $0 < p < Q$, we say that K is a homogeneous convolution operator of type p . As we mentioned before, a homogeneous function is in L^p_{loc} so the integral in (11) exists for $f \in C_c^\infty(\mathbf{F}^m)$.

Suppose k is homogeneous of degree $-Q$ and for each entry

$$k_{ij}, \quad 1 \leq i \leq m', \quad 1 \leq j \leq m ,$$

we have

$$(12) \quad \int_{a \leq |x| \leq b} k_{ij}(x)dx = 0$$

for all a and b . We say an operator K is of type 0 if for some constant c we have

$$Kf(x) = \lim_{\epsilon \rightarrow 0} \int_{\epsilon \leq |y| \leq 1/\epsilon} k(y^{-1}x)f(y)dy + cf(0) \quad \text{for all } f \in C_c^\infty(\mathbf{F}^m)$$

where k satisfies (12). We refer the reader to Folland [9] or Rothschild and Stein [16] for details.

To study the continuity properties of these operators we define L^p spaces and Sobolev-type spaces of sections from N to \mathbf{F}^m . Let $\| \cdot \|_{L^p}$ denote the usual L^p norm on functions. Let $f \in C_c^\infty(\mathbf{F}^m)$ and let $f_i, i = 1, \dots, m$ be the components of f . Define the norm

$$\| f \|_{L^p(\mathbf{F}^m)} = \left(\sum_{i=1}^m \| f_i \|_{L^p}^p \right)^{1/p} .$$

Let $L^p(\mathbf{F}^m)$ be the completion of $C_c^\infty(\mathbf{F}^m)$ under this norm.

Let $\{X_{1,1}, \dots, X_{1,d}\}$ be the orthonormal basis of \mathfrak{n}_1 , with $d = \dim(\mathfrak{n}_1)$. For brevity, we will drop reference to the first subscript. Let J be a multi-index, $J = (j_1, j_2, \dots, j_q)$ with $1 \leq j_1 < j_2 < \dots < j_q \leq d$. Define $|J| = q$ and define $X_J = X_{j_1}X_{j_2} \dots X_{j_q}$. Define $S_q^p(\mathbf{F}^m)$ to be the closure of $C_c^\infty(\mathbf{F}^m)$ under the norm

$$\| f \|_{S_q^p(\mathbf{F}^m)} = \left(\| f \|_{L^p(\mathbf{F}^m)}^p + \sum_{i=1}^m \sum_{|J| \leq q} \| X_J f_i \|_{L^p}^p \right)^{1/p} .$$

A modification of a theorem by Folland [9] yields

THEOREM 4. (i) Let K be a convolution operator of type r for $r > 0$. Then K extends from $C_c^\infty(\mathbf{F}^m)$ to a bounded operator from $L^p(\mathbf{F}^m)$ to $L^q(\mathbf{F}^m)$ where $1 < p < Q/r$ and $q^{-1} = p^{-1} - r/Q$. (ii) Let K be a convolution operator of type 0. Then K extends from $C_c^\infty(\mathbf{F}^m)$ to a bounded operator from $S_k^p(\mathbf{F}^m)$ to $S_k^p(\mathbf{F}^m)$.

Finally, we mention the interaction between the homogeneous convolution operators and the left-invariant differential operators. Let $D: C^\infty(\mathbf{F}^m) \rightarrow C^\infty(\mathbf{F}^m)$ be a left-invariant homogeneous differential operator of degree 1 and let K be a homogeneous convolution operator of type r , with $r \geq 1$. Then DK is a homogeneous convolution operator of type $r - 1$. Moreover, if $r > 1$ the kernel of DK is given by $Dk(x)$.

4. THE HODGE DECOMPOSITION

Consider the complex (1) where $E_i = \mathbf{R}^n \times \mathbf{F}^{m_i}$. Assume that each of the D_i is a first order, left-invariant operator, homogeneous of degree 1. So each entry of D_i is of the form $\sum_{j=1}^d a_j X_{1,j}$ where a_j is constant. Construct the Laplacian, Δ , with respect to the euclidian inner products on \mathbf{F}^{m_i} , $i = 1, 2, 3$. Assume there exists a homogeneous convolution operator of type 2, K , which inverts Δ . If $f \in C_c^\infty(\mathbf{F}^{m_2})$ then $f(x) = \Delta K f(x) = K \Delta f(x)$.

THEOREM 5. Let $f \in S_2^2(\mathbf{F}^{m_2})$. As distributions, $\Delta f = 0$ if and only if $f = 0$.

Proof. Obviously, if $f = 0$ then $\Delta f = 0$.

Assume $\Delta f = 0$. Let $\{f_j\}$ be a sequence in $C_c^\infty(\mathbf{F}^{m_2})$ such that $f_j \rightarrow f$ in $S_2^2(\mathbf{F}^{m_2})$. Then $f_j \rightarrow f$ in the sense of distributions. Moreover, $\Delta f_j \rightarrow \Delta f = 0$ in $L^2(\mathbf{F}^{m_2})$. Let $g \in C_c^\infty(\mathbf{F}^{m_2})$. Then

$$\langle f, g \rangle = \lim_{j \rightarrow \infty} \langle f_j, g \rangle = \lim_{j \rightarrow \infty} \langle f_j, \Delta K g \rangle = \lim_{j \rightarrow \infty} \langle \Delta f_j, K g \rangle.$$

Because $g \in C_c^\infty(\mathbf{F}^{m_2})$ it is in L^p where $p = 2Q/(Q+4)$. Therefore, by Theorem 4(i), $Kg \in L^q$ where

$$q^{-1} = (Q+4)/2Q - 2/Q = 1/2, \text{ i.e., } Kg \in L^2(\mathbf{F}^{m_2}).$$

For $Q \geq 5$, $1 < p < q < \infty$. So

$$|\langle f, g \rangle| = \lim_{j \rightarrow \infty} |\langle \Delta f_j, K g \rangle| \leq \lim_{j \rightarrow \infty} \|\Delta f_j\|_{L^2(\mathbf{F}^{m_2})} \|Kg\|_{L^2(\mathbf{F}^{m_2})} = 0.$$