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Autor: Duddy, John
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Let H be the kernel of Δ in $C_c^\infty(E_2)$. A Hodge decomposition for $C_c^\infty(E_2)$ is

$$C_c^\infty(E_2) = D_1(C_c^\infty(E_1)) \oplus D_2^*(C_c^\infty(E_3)) \oplus H.$$

Hodge studied the de Rham complex on a compact Riemannian manifold. The Riemannian metric induced the metrics on the bundles $\Lambda^p T^*(M)$ as well as the volume element.

In the next section we discuss abstract CR manifolds and look at the Heisenberg group in detail. We write down the $\bar{\partial}_b$ and \square_b operators explicitly and give Folland and Stein's inverse to \square_b . In section 3 we introduce the stratified Lie groups and the associated homogeneous structures. We present the continuity theorems of Folland and Rothschild and Stein for convolution operators. In section 4 we prove the decomposition theorem in the general setting of stratified groups.

These results are an extension of the author's dissertation [4]. We wish to express our deep gratitude to M. Kuranishi. We would also like to thank D. Tartakoff for his help and encouragement.

2. CR STRUCTURES AND THE HEISENBERG GROUP

Let M be a C^∞ manifold of dimension $2n + 1$. The complexified tangent bundle of M , $CT(M)$, is the bundle whose fiber is $\mathbb{C} \otimes_{\mathbb{R}} T_m(M)$ where $T_m(M)$ is the tangent space at $m \in M$. When there is no confusion we will drop reference to M in the notation for $T(M)$, $CT(M)$, etc. So, $T(M) = T$ and $CT(M) = CT$, for example.

A CR structure on M is a subbundle $T_{1,0} \subset CT$ such that (i) $T_{1,0} \cap \overline{T_{1,0}} = \{0\}$, (ii) $\text{codim}(T_{1,0} \otimes \overline{T_{1,0}}) = 1$, (iii) if X and Y are smooth sections of $T_{1,0}$ then $[X, Y] = XY - YX$ is a section $T_{1,0}$. We set $T_{0,1} = \overline{T_{1,0}}$. If M has a CR structure it is called a CR manifold.

An example of a CR manifold is a real hypersurface M in a complex manifold M' , $M \subset M'$. Define $T_{1,0}(M) = CT(M) \cap T_{1,0}(M')$ where $T_{1,0}(M')$ is the holomorphic tangent bundle of M' .

If M is a CR manifold set $T^{1,0}$ (resp., $T^{0,1}$) to be the dual space to $T_{1,0}$ (resp., $T_{0,1}$). Let $\Lambda^{p,q}$ be the space of C^∞ sections of $\Lambda^p T^{1,0} \otimes \Lambda^q T^{0,1}$. Define the operator $\bar{\partial}_b: \Lambda^{p,q} \rightarrow \Lambda^{p,q+1}$ as follows: Let $\phi \in \Lambda^{p,q}$ and let X_1, \dots, X_p (resp., Y_1, \dots, Y_{q+1}) be sections of $T_{1,0}$ (resp., $T_{0,1}$). Then

$$\begin{aligned} & \langle \bar{\partial}_b \phi; (X_1 \wedge \dots \wedge X_p) \otimes (Y_1 \wedge \dots \wedge Y_{q+1}) \rangle \\ &= (q+1)^{-1} \sum_{j=1}^{q+1} (-1)^{j+1} Y_j \langle \phi; (X_1 \wedge \dots \wedge X_p) \otimes (Y_1 \wedge \dots \wedge \hat{Y}_j \dots \wedge Y_{q+1}) \rangle \\ + & (q+1)^{-1} \sum_{i < j} (-1)^{i+j} \langle \phi; (X_1 \wedge \dots \wedge X_p) \otimes ([Y_i, Y_j] \wedge Y_1 \wedge \dots \wedge \hat{Y}_i \dots \wedge \hat{Y}_j \dots \wedge Y_{q+1}) \rangle. \end{aligned}$$

The $\hat{}$ symbol over a section means as usual that it is deleted from the expression. One can show that

- i) $\bar{\partial}_b^2 = 0$,
- ii) $\bar{\partial}_b(\phi \wedge \psi) = (\bar{\partial}_b \phi) \wedge \psi + (-1)^p \phi \wedge \bar{\partial}_b \psi$ for $\phi \in \Lambda^{0,p}$,
- iii) $\langle \bar{\partial}_b f, Y \rangle = Yf$ for $f \in \Lambda^{0,0}$ and Y a section of $T_{0,1}$.

See [6] for details.

The Heisenberg group, H , is a Lie group with a natural CR structure. The manifold is $\mathbf{C}^n \times \mathbf{R}$. Let $(z, t), (z', t') \in \mathbf{C}^n \times \mathbf{R} = H$. The group law is defined by

$$(z, t) \cdot (z', t') = (z + z', t + t' + 2 \operatorname{Im}(z \cdot z'))$$

where $z \cdot z' = \sum_{j=1}^n z_j \bar{z}'_j$. The identity element is $(0, 0)$ and $(z, t)^{-1} = (-z, -t)$.

Sometimes we will set $u = (z, t)$.

For $j = 1, \dots, n$ if we set $z_j = x_j + iy_j$, the mapping

$$(z, t) \rightarrow (x_1, \dots, x_n, y_1, \dots, y_n, t)$$

defines a C^∞ coordinate system on H . The left invariant vector fields (i.e., the elements of the Lie algebra) are \mathbf{R} -linear combinations of

$$(2) \quad X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \quad T = \frac{\partial}{\partial t}.$$

They satisfy the following commutation relations:

$$(3) \quad [X_j, T] = [Y_j, T] = [X_j, X_k] = [Y_j, Y_k] = 0,$$

$$(4) \quad [X_j, Y_k] = -4\delta_{jk}T.$$

Let CT be the complexified tangent bundle of H . Define

$$(5) \quad Z_j = \frac{1}{2}(X_j - iY_j) = \frac{\partial}{\partial z_j} + i\bar{z}_j \frac{\partial}{\partial t}.$$

Then $Z_j, \bar{Z}_j,$ and T form a basis (over \mathbf{R}) of the space of left invariant complex tangent vector fields. In particular, they form a global frame of CT . From (3), (4) and (5) we easily see that

$$(6) \quad [Z_j, Z_k] = [\bar{Z}_j, \bar{Z}_k] = [Z_j, T] = [\bar{Z}_j, T] = 0$$

$$[Z_j, \bar{Z}_k] = -2i\delta_{jk}T.$$

Let $T_{1,0}$ (resp., $T_{0,1}$) be the subbundle of CT spanned by the Z_j 's (resp., \bar{Z}_j 's). Then

$$\bar{T}_{1,0} = T_{0,1}$$

$$T_{1,0} \cap T_{0,1} = \{0\}$$

$$\text{codim}(T_{1,0} \otimes T_{0,1}) = 1.$$

Also, if V_1, V_2 are sections of $T^{1,0}$ we can write $V_i = \sum_{j=1}^n f_{ij}Z_j, i = 1, 2$ where the f_{ij} are C^∞ functions on H . Then by (6),

$$[V_1, V_2] = \sum_{k=1}^n \left(\sum_{j=1}^n (f_{1j}Z_j f_{2k} - f_{2j}Z_j f_{1k}) \right) Z_k.$$

So, the splitting of CT defines a CR structure on H .

Impose the left invariant Hermitian metric on CT which makes the Z 's, \bar{Z} 's and T an orthonormal frame. Let ω^j and τ be dual to Z_j and T , respectively. Then $\omega^j, \bar{\omega}^j$ and τ form an orthonormal frame for CT^* . The volume element on H is

$$(7) \quad du = 2^n dx_1 \wedge \dots \wedge dx_n \wedge dy_1 \wedge \dots \wedge dy_n \wedge dt.$$

Since H is nilpotent and, hence, unimodular, the volume element is both left and right invariant. One can also verify this directly.

Let $J = (j_1, \dots, j_q)$ be a multi-index with $1 \leq j_i \leq n, i = 1, \dots, q$. Define $|J| = q$ and $\bar{\omega}^J = \bar{\omega}^{j_1} \wedge \dots \wedge \bar{\omega}^{j_q}$. If $\phi \in \Lambda^{0,q}(H)$ we may write $\phi = \sum_{|J|=q} \phi_J \bar{\omega}^J$ where ϕ_J is a C^∞ function from H to \mathbf{C} . Let

$$\bar{\omega}^j \lrcorner \bar{\omega}^J = (-1)^k \bar{\omega}^{j_1} \wedge \dots \wedge \widehat{\bar{\omega}^{j_k}} \wedge \dots \wedge \bar{\omega}^{j_q} \quad \text{if } j = j_k \quad \text{and } \bar{\omega}^j \lrcorner \bar{\omega}^J = 0$$

otherwise. Folland and Stein prove that for $\phi \in \Lambda^{0,q}$

$$i) \quad \bar{\partial}_b \phi = \sum_{|J|=q} \sum_{j=1}^n \bar{Z}_j \phi_J \bar{\omega}^j \wedge \bar{\omega}^J,$$

$$(8) \quad \text{ii) } \bar{\partial}_b^* \phi = - \sum_{|J|=q} \sum_{j=1}^n Z_j \phi_J \bar{\omega}^j \lrcorner \bar{\omega}^J,$$

$$\text{iii) } \square_b \phi = \sum_{|J|=q} \left(-\frac{1}{2} \sum_{j=1}^n (Z_j \bar{Z}_j + \bar{Z}_j Z_j) + i(n-2q)T \right) \phi_J \bar{\omega}^J.$$

Define the function

$$\Phi_\alpha(z, t) = (|z|^2 - it)^{-\frac{n+\alpha}{2}} (|z|^2 + it)^{-\frac{n-\alpha}{2}}.$$

Let $\phi \in \Lambda_c^{0,q}$, $q \neq 0$, n . For an appropriate constant, c_q , define

$$(9) \quad K_q \phi(v) = c_q \sum_{|J|=q} \left(\int_H \phi_J(u) \Phi_{n-2q}(u^{-1}v) du \right) \bar{\omega}^J.$$

Folland and Stein prove that for the appropriate c_q

THEOREM 1. *Let $\phi \in \Lambda_c^{0,q}$, $q \neq 0$, n . Then $\square_b K_q \phi = K_q \square_b \phi = \phi$.*

In [4] we prove a stronger version of the following Hodge decomposition theorem.

THEOREM 2. *Let $\phi \in \Lambda_c^{0,q}$, $q \neq 0$, n . Then*

- i) $H\phi = 0$ where H is the orthogonal projection onto the kernel of \square_b .
- ii) $\phi = \bar{\partial}_b \bar{\partial}_b^* K_q \phi + \bar{\partial}_b^* \bar{\partial}_b K_q \phi$.

We also prove

THEOREM 3. *If $\phi \in \Lambda_c^{0,q}$, $q \neq 0$, n and if $\bar{\partial}_b \phi = 0$ then $\psi = \bar{\partial}_b^* K_q \phi$ satisfies $\bar{\partial}_b \psi = \phi$.*

These two theorems are special cases of theorems 6 and 7 proven in section 4.

3. DIFFERENTIAL COMPLEXES ON STRATIFIED GROUPS

We study a class of nilpotent Lie groups which we describe in terms of their Lie algebras. A graded Lie algebra, \mathfrak{n} , is a finite dimensional nilpotent algebra which has a direct sum decomposition, $\mathfrak{n} = \bigoplus_{i=1}^r \mathfrak{n}_i$ where the \mathfrak{n}_i satisfy

- i) $[\mathfrak{n}_i, \mathfrak{n}_j] \subseteq \mathfrak{n}_{i+j}$ if $i + j \leq r$,
- ii) $[\mathfrak{n}_i, \mathfrak{n}_j] = 0$ if $i + j > r$.