The only identifications under the action are: vu gets identified to vx and wu gets identified to wx. It follows that $P/T(r, s, t)$ is the 2 sphere and the branched covering $P/\Delta \to P/T(r, s, t)$ has 3 branch points coming from the vertices u, v, w.

Now notice that Δ is torsion free. This follows from the facts:

(1) the elements of finite order in $T(r, s, t)$ are the conjugates of A, B, C.

(2) elements of finite order in $T(r, s, t)$ map to elements of the same order in G. From this it follows that the orders of the branch points are r, s, t respectively.

Finally we consider the Riemann-Hurwitz formula:

$$\chi(P/\Delta) = |G| \left(\chi(P/T(r, s, t)) - \left(1 - \frac{1}{r}\right) - \left(1 - \frac{1}{s}\right) - \left(1 - \frac{1}{t}\right) \right)$$

i.e.,
$$2 - 2g = |G| \left(\frac{1}{r} + \frac{1}{s} + \frac{1}{t} - 1 \right).$$

Therefore
$$g = 1 + \frac{|G|}{2} \left(1 - \frac{1}{r} - \frac{1}{s} - \frac{1}{t} \right)$$
Q.e.d.

§ 3. Conformal Actions on Surfaces of least Genus

If (A, B, C) is an (r, s, t) triple generating $PSl_2(p)$ then we have a short exact sequence

$$1 \to \Delta \to T(r, s, t) \to PSl_2(p) \to 1$$

where Δ is torsion free. Then it follows that $H/T(r, s, t)$ is S^2 and the branched covering $H/\Delta \to H/T(r, s, t)$ has 3 branch points with orders r, s, t.

Conversely we have:

(3.1). Theorem. *If S is a Riemann surface of least genus for $PSl_2(p)$ then $S/PSl_2(p)$ is S^2 and $\pi: S \to S/PSl_2(p)$ has 3 branch points.*

Proof. There exists a short exact sequence $1 \to \Delta \to T(2, 3, p) \to PSl_2(p) \to 1$ arising from a $(2, 3, p)$ triple and consequently

$$\text{genus}\,(H/\Delta) = 1 + \frac{|G|}{2} \left(\frac{1}{6} - \frac{1}{p} \right).$$

Let g = genus (S), h = genus $(S/PSl_2(p))$ and suppose $\pi: S \to S/PSl_2(p)$ has b branch points $x_1, ..., x_b$ of respective orders $n_1, ..., n_b$. Then the Riemann-Hurwitz formula tells us

$$(3.2). \qquad 2 - 2g = |G|\left(2 - 2h - \sum\left(1 - \frac{1}{n_i}\right)\right).$$

That is $g = 1 + \dfrac{|G|}{2}\left(2h - 2 + \sum\left(1 - \dfrac{1}{n_i}\right)\right)$. Since g is the least genus this leads to the inequality

$$(3.3). \qquad 2h - 2 + \sum\left(1 - \frac{1}{n_i}\right) \leqslant \frac{1}{6} - \frac{1}{p}.$$

From this we immediately see that $h = 0, 1$.

Therefore we suppose that $h = 1$. Since all $n_i \geqslant 2$ this implies that $b = 0$, and hence $PSl_2(p)$ is acting fixed point freely on S with orbit space the torus. But this immediately gives an epimorphism $\mathbf{Z} \oplus \mathbf{Z} \twoheadrightarrow PSl_2(p)$. However, this is a contradiction since G is not abelian. Therefore $h = 0$ and $S/PSl_2(p)$ is a 2-sphere.

To prove that there are 3 branch points put $h = 0$ into (3.3):

$$(3.4) \qquad -2 + \sum_{i=1}^{b}\left(1 - \frac{1}{n_i}\right) \leqslant \frac{1}{6} - \frac{1}{p}.$$

Since $1 - \dfrac{1}{n_i} \geqslant \dfrac{1}{2}$ for all i this gives $b \leqslant 4$. If $b = 0$ we have an unbranched covering $S \to S^2$ with deck transformation group $PSl_2(p)$. But this is clearly a contradiction.

Thus assume $b = 1$. Then we have the regular unbranched covering

$$S - \pi^{-1}(x_1) \to S^2 - \{x_1\}$$

with deck transformation group $PSl_2(p)$. But again this is impossible since $S^2 - \{x_1\} \cong \mathbf{R}^2$.

Next we put $b = 2$ and consider the regular covering

$$S - \pi^{-1}\{x_1, x_2\} \to S^2 - \{x_1, x_2\}.$$

Then we have the exact sequence coming from fundamental groups $1 \to \mathbf{Z} \to \mathbf{Z} \to PSl_2(p) \to 1$, which is again a contradiction.

Finally we suppose $b = 4$. The inequality (3.4) is

(3.5).
$$2 - \frac{1}{n_1} - \frac{1}{n_2} - \frac{1}{n_3} - \frac{1}{n_4} \leqslant \frac{1}{6} - \frac{1}{p}$$

and is clearly satisfied by $n_1 = n_2 = n_3 = n_4 = 2$. However, this choice of n_i's gives $g = 1$ by (3.2); in other words $PSl_2(p)$ is toroidal. However, no nonabelian finite simple group G can act on $S^1 \times S^1$ because covering space theory implies there are branch points and hence the orbit space is S^2. Hence the induced homomorphism $PSl_2(p) \to \mathrm{Aut}(\mathbf{Z}^2)$ is nontrivial and also has a kernel since $\mathrm{Aut}(\mathbf{Z}^2)$ has no p torsion for $p \geqslant 7$. This contradicts G simple. Therefore this case is excluded and we have

$$2 - \frac{1}{n_1} - \frac{1}{n_2} - \frac{1}{n_3} - \frac{1}{n_4} \geqslant 2 - \frac{1}{2} - \frac{1}{2} - \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

contradicting (3.5). Q.e.d.

If we let the orders of the 3 branch points be r, s, t then the Riemann-Hurwitz formula is

$$\chi(S) = |PSl_2(p)| \left(2 - \left(1 - \frac{1}{r}\right) - \left(1 - \frac{1}{s}\right) - \left(1 - \frac{1}{t}\right)\right).$$

But $|PSl_2(p)| = \dfrac{p(p^2-1)}{2}$ and therefore

$$\text{genus}\ (S) = 1 + \frac{p(p^2-1)}{4}\left(1 - \frac{1}{r} - \frac{1}{s} - \frac{1}{t}\right).$$

To take advantage of this formula we must know what sort of branching data (r, s, t) can occur. To this end we quote a very general theorem of Tucker [T].

(3.6). THEOREM. *Suppose G is a finite group acting effectively on a closed orientable surface S by orientation preserving homeomorphisms. If $g = \text{genus}\,(S/G)$ and there are b branch points of orders $n_1, \ldots, n_b$ then G has a presentation of the form*

$$\{x_1, y_1, \ldots, x_g, y_g, e_1, \ldots, e_b \mid \prod_{i=1}^{g} [x_i, y_i] e_1 \ldots e_b = e_1^{n_1} = \ldots = e_b^{n_b} = 1, ETC\}$$

(3.7). COROLLARY. *If S is a Riemann surface of least genus for $PSl_2(p)$ then there exist integers $r, s, t, \geqslant 2$ so that*

(a) *there is an extension $1 \to \Delta \to T(r, s, t) \to PSl_2(p) \to 1$;*

(b) $\text{genus}\,(PSl_2(p)) = 1 + \frac{p(p^2-1)}{4}\left(1 - \frac{1}{r} - \frac{1}{s} - \frac{1}{t}\right).$

If A, B, C are the usual generators of $T(r, s, t)$ then it is in fact true that the orders of A, B, C in $PSl_2(p)$ are r, s, t. Putting (2.22) and (3.7) together gives

(3.8). COROLLARY. *The genus of* $PSl_2(p)$ *is given by*

$$g = \min\left\{1 + \frac{p(p^2-1)}{4}\left(1 - \frac{1}{r} - \frac{1}{s} - \frac{1}{t}\right)\right\}$$

where the minimum is taken over all (r, s, t) *for which there exist* (r, s, t) *triples generating* $PSl_2(p)$.

The last step in the determination of the genus is to identify those (r, s, t) which are relevant. This is accomplished in the following manner:

(1) first find all (r, s, t) so that

$$1 - \frac{1}{r} - \frac{1}{s} - \frac{1}{t} \leqslant \frac{1}{6} - \frac{1}{d}, \text{ assuming } p \geqslant 13\,.$$

(2) then eliminate those triples (r, s, t) corresponding to either spherical or Euclidean triangle groups.

(3) make a comparison of the triples remaining so as to eliminate those with larger genus.

In the following table we give some pertinent data:

TABLE I

(r, s, t)	$1 - \frac{1}{r} - \frac{1}{s} - \frac{1}{t}$	type	condition for $1 - \frac{1}{r} - \frac{1}{s} - \frac{1}{t} \leqslant \frac{1}{6} - \frac{1}{d}$
$(2, 2, t)$	$-\frac{1}{t}$	spherical	$d \geqslant 6$
$(2, 3, t)$ where $3 < t < 5$	$\frac{1}{6} - \frac{1}{t}$	spherical	$t \leqslant d$
$(2, 3, 6)$	0	euclidean	$t < d$
$(2, 3, t)$ where $t > 7$	$\frac{1}{6} - \frac{1}{t}$	hyperbolic	$t \leqslant d$
$(2, 4, 4)$	0	euclidean	$d > 6$
$(2, 4, 5)$	$\frac{1}{20}$	hyperbolic	$d \geqslant 9$
$(2, 4, 6)$	$\frac{1}{12}$	hyperbolic	$d \geqslant 12$
$(2, 4, 7)$	$\frac{3}{28}$	hyperbolic	$d \geqslant 17$
$(2, 4, 8)$	$\frac{1}{8}$	hyperbolic	$d \geqslant 24$
$(2, 4, 9)$	$\frac{5}{36}$	hyperbolic	$d \geqslant 36$
$(2, 4, 10)$	$\frac{3}{20}$	hyperbolic	$d \geqslant 60$
$(2, 4, 11)$	$\frac{7}{44}$	hyperbolic	$d \geqslant 132$

TABLE I *(suite)*

	(r, s, t)	$1 - \frac{1}{r} - \frac{1}{s} - \frac{1}{t}$	*type*	condition for $1 - \frac{1}{r} - \frac{1}{s} - \frac{1}{t} \leqslant \frac{1}{6}$
13.	$(2, 4, t)$	$\frac{1}{4} - \frac{1}{t} \geqslant \frac{1}{6}$	hyperbolic	never
14.	$(2, 5, 5)$	$\frac{1}{10}$	hyperbolic	$d \geqslant 15$
15.	$(2, 5, 6)$	$\frac{2}{15}$	hyperbolic	$d \geqslant 30$
16.	$(2, 5, 7)$	$\frac{11}{70}$	hyperbolic	$d \geqslant 105$
17.	$(2, 5, t)$ where $t > 8$	$\frac{3}{10} - \frac{1}{t} > \frac{1}{6}$	hyperbolic	never
18.	$(2, s, t)$	$\frac{1}{2} - \frac{1}{s} - \frac{1}{t} \geqslant \frac{1}{6}$	hyperbolic	never
19.	$(3, 3, 3)$	0	euclidean	$d > 7$
20.	$(3, 3, 4)$	$\frac{1}{12}$	hyperbolic	$d \geqslant 12$
21.	$(3, 3, 5)$	$\frac{2}{15}$	hyperbolic	$d \geqslant 30$
22.	$(3, 3, t)$	$\frac{1}{3} - \frac{1}{t} \geqslant \frac{1}{6}$	hyperbolic	never
23.	$(3, s, t)$ where $t > s > 4$	$\frac{2}{3} - \frac{1}{s} - \frac{1}{t} \geqslant \frac{1}{6}$	hyperbolic	never
24.	(r, s, t) where $t > s > r > 4$	$1 - \frac{1}{r} - \frac{1}{s} - \frac{1}{t} \geqslant \frac{1}{4}$	hyperbolic	never

Examining this table we see that we can eliminate cases 13, 17, 18, 22, 23 and 24 since $\frac{1}{6} - \frac{1}{d}$ will always be less than $1 - 1/r - 1/s - 1/t$. We can also eliminate cases 1, 2, 3, 5 and 19 since these triples are not hyperbolic. Now notice that cases 7, ..., 12 need never be considered since if there are such triples generating $PSl_2(p)$ then there will also be a (3, 3, 4) triple generating $PSl_2(p)$, in which case the genus calculation from the (3, 3, 4) case is at least as small. In a similar fashion we can ignore cases 15, 16 and 21 by comparing them with case 14. Finally, we can use Lemma (2.3) to eliminate case 4. The triples remaining after this will be $(2, 3, p)$, $(2, 3, d)$, (2, 5, 5), (2, 4, 5) and (3, 3, 4). Minimization of the genera for these triples leads directly to the corollary in the introduction.

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Henry Glover

Ohio State University
Columbus, OH 43210

Denis Sjerve

University of British Columbia
Vancouver, BC V6T1W5