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times, namely for $p = 103, 137$ and 569 and $(2, 4, 5)$ occurs exactly six times, for $p = 199, 239, 359, 439, 521$ and 599 .

If $S = H/\Delta$ is the surface of minimal genus for $PSl_2(p)$ coming from one of the extensions above then the orbit manifold $S/PSl_2(p)$ is the 2-sphere S^2 and the quotient map $S \rightarrow S^2$ is a branched covering with exactly 3 branch points. One of the most important steps in the proof of the main result of this paper is the converse, namely if S is a Riemann surface of least genus for the group $G = PSl_2(p)$ then $S/G = S^2$ and $S \rightarrow S^2$ is a branched covering with exactly 3 branch points (see section 3). Note that a related notion of genus, "the Cayley genus of a group" has been studied by others, among them Tucker [T]. Earlier results can be found in Hurwitz [H] and Burnside [B].

The remainder of this paper is organized as follows. In section 2 we describe various ways of generating $PSl_2(p)$ and then prove theorems I and II. Section 3 proves that if S is a Riemann surface of least genus for $PSl_2(p)$ then $S/PSl_2(p)$ is a 2-sphere S^2 and the branched covering $S \rightarrow S^2$ has exactly 3 branch points. The calculation of genus ($PSl_2(p)$) then follows from the results of section 2.

Finally we would like to thank Bomshik Chang for help with the group theory of $PSl_2(p)$. The first author would like to thank the University of British Columbia for its hospitality to him during the time this research was done.

§ 2. GENERATING TRIPLES FOR $PSl_2(p)$

Our goal in this section is to find triples (r, s, t) for which there are epimorphisms $T(r, s, t) \rightarrow PSl_2(p)$. In other words, given integers $r, s, t \geq 2$ are there matrices $A, B, C \in PSl_2(p)$ so that A, B, C generate $PSl_2(p)$ and $A^r = B^s = C^t = ABC = 1$? Throughout this section a standard reference for the group theory is Suzuki [S].

The spherical triangle groups are given in the following table

TABLE I

<i>triple</i>	<i>triangle group</i>	<i>order</i>
$(2, 2, n)$	dihedral	$2n$
$(2, 3, 3)$	tetrahedral (A_4)	12
$(2, 3, 4)$	octahedral (S_4)	24
$(2, 3, 5)$	icosahedral (A_5)	60

Now the group $PSl_2(p)$ has an element of order p since its order is $|PSl_2(p)| = \frac{p(p^2-1)}{2}$. It therefore follows that $PSl_2(p)$ is not the image of any spherical triangle group since $PSl_2(p)$ can not be the image of any dihedral group and we are assuming $p \geq 7$. The following lemma then implies that $PSl_2(p)$ can only be the image of hyperbolic triangle groups.

(2.1). LEMMA. $PSl_2(p)$ is not the image of any euclidean triangle group.

Proof. Suppose T is one of the euclidean triangle groups, namely one of $T(3, 3, 3)$, $T(2, 4, 4)$, $T(2, 3, 6)$, and there exists an epimorphism $T \rightarrow PSl_2(p)$. Since T has $\mathbf{Z} \oplus \mathbf{Z}$ as a normal subgroup of index ≤ 6 it follows that $PSl_2(p)$ has an abelian normal subgroup of index ≤ 6 . But this is clearly not possible. Q.e.d.

In order to decide when a triple of matrices $A, B, C \in PSl_2(p)$ generates the entire group we need detailed knowledge of the maximal subgroups. The following theorem can be found in Suzuki [S].

(2.2). THEOREM. The maximal proper subgroups of $PSl_2(p)$ are:

- (a) dihedral of order $p - 1$ or $p + 1$.
- (b) solvable of order $\frac{p(p-1)}{2}$.
- (c) A_4 if $p \equiv 3, 13, 27, 37 \pmod{40}$.
- (d) S_4 if $p \equiv \pm 1 \pmod{8}$.
- (e) A_5 if $p \equiv \pm 1 \pmod{5}$.

The dihedral group of order $p - 1$ can be chosen to be

$$D = \langle R, S \rangle = \left\{ \left[\begin{array}{cc} \alpha & 0 \\ 0 & \alpha^{-1} \end{array} \right], \left[\begin{array}{cc} 0 & \alpha \\ -\alpha^{-1} & 0 \end{array} \right] \mid \alpha \in \mathbf{Z}_p^* \right\}, \quad \text{where}$$

$$R = \left[\begin{array}{cc} x & 0 \\ 0 & x^{-1} \end{array} \right], \quad S = \left[\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right] \quad \text{and } x \text{ is a primitive root mod } p.$$

To realize the dihedral subgroup of order $p + 1$ we need another description of $PSl_2(p)$. The mapping

$$GF(p^2) \rightarrow GF(p^2), \quad x \rightarrow x^p$$

is an automorphism of order 2. For convenience we put $\bar{x} = x^p$. Then $PSl_2(p) \cong PSU_2(p)$, where $PSU_2(p)$ is the projective special unitary group

$$PSU_2(p) = \left\{ \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \mid a, b \in GF(p^2), a\bar{a} + b\bar{b} = 1 \right\}$$

Now consider the matrix $U = \begin{bmatrix} \omega & 0 \\ 0 & \bar{\omega} \end{bmatrix}$, where $\omega \in GF(p^2)$ is chosen so that $\omega^{(p+1)/2} = -1$ and $\omega^k \neq \pm 1$ for $1 \leq k < \frac{p+1}{2}$. Then the order of U as an element of $PSU_2(p)$ is $\frac{p+1}{2}$ and the dihedral group of order $\frac{p+1}{2}$ can be taken to be

$$D = \langle U, S \rangle = \left\{ \begin{bmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{bmatrix}, \begin{bmatrix} 0 & \alpha \\ -\bar{\alpha} & 0 \end{bmatrix} \mid \alpha \in GF(p^2)^*, \alpha^{p+1} = 1 \right\}.$$

Finally the maximal solvable subgroup of order $\frac{p(p-1)}{2}$ can be chosen to be the subgroup of upper triangular matrices

$$H = \left\{ \begin{bmatrix} x & \lambda \\ 0 & x^{-1} \end{bmatrix} \mid x \in \mathbf{Z}_p^*, \lambda \in \mathbf{Z}_p \right\}.$$

Thus there is a split extension of the form

$$1 \rightarrow \mathbf{Z}_p \rightarrow H \xrightarrow{\theta} \mathbf{Z}_{(p-1)/2} \rightarrow 1, \theta: \begin{bmatrix} x & \lambda \\ 0 & x^{-1} \end{bmatrix} \rightarrow \pm x.$$

The kernel is generated by $T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and the splitting is induced by the matrix $\begin{bmatrix} x & 0 \\ 0 & x^{-1} \end{bmatrix}$, where x is a primitive root mod p .

The other maximal subgroups will not play much of a role in what follows. Notice that an immediate consequence of (2.2) is

(2.3). LEMMA.

(a) *The order of an element of $PSl_2(p)$ is one of the following: a divisor of either $\frac{p-1}{2}$ or $\frac{p+1}{2}$; p ; 2, 3, 4 or 5.*

(b) If d is a divisor of either $\frac{p-1}{2}$ or $\frac{p+1}{2}$ then there is an element of $PSl_2(p)$ having order d .

The order of an element $A \in PSl_2(p)$ can be determined from its trace. In particular we have:

(2.4) LEMMA. Let $A \in PSl_2(p)$ and $\chi = \pm \text{trace } A$. Then the order of A is 2, 3, 4, or 5 respectively if, and only if, $\chi \equiv 0 (p)$, $\chi \equiv \pm 1 (p)$, $\chi^2 \equiv 2 (p)$ or $\chi^2 \pm \chi - 1 \equiv 0 (p)$ respectively.

Definition. We say that a triple of elements (A, B, C) from $PSl_2(p)$ is an (r, s, t) triple if (a) order $A = r$, order $B = s$, order $C = t$; and (b) $ABC = 1$.

In order to construct $(2, 3, d)$ triples for $d \mid \frac{p-1}{2}$ let A, B, C be the matrices

$$(2.5) \quad A = \begin{bmatrix} 0 & -x \\ x^{-1} & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}, \quad C = (AB)^{-1} = \begin{bmatrix} x^{-1} & x \\ 0 & x \end{bmatrix}$$

where $x \in \mathbf{Z}_p^*$. Then order $A = 2$, order $B = 3$ and

$$C^k = \begin{bmatrix} x^{-k} & x(x^{k-1} + x^{k-3} + \dots + x^{-(k-1)}) \\ 0 & x^k \end{bmatrix}$$

If $x = \pm 1$ then $C = T$ and order $T = p$. In general the order of C is given by the following lemma whose proof is elementary and hence omitted.

(2.6). LEMMA. Assume $x \neq \pm 1$. Then the order of C in $PSl_2(p)$ is the least positive integer k so that either $x^k = 1$ or $x^k = -1$.

Given $x \in \mathbf{Z}_p^*$, $x \neq \pm 1$, let k be the least positive integer so that $x^k = \pm 1$. Since we always have $x^{(p-1)/2} = \pm 1$ it follows that $1 < k \leq \frac{p-1}{2}$. Also $x^{2k} = 1$ and therefore $k \mid \frac{p-1}{2}$. Conversely, given any divisor d of $\frac{p-1}{2}$ there exists $x \in \mathbf{Z}_p^*$ so that d is the least positive integer k satisfying $x^k = \pm 1$.

(2.7). COROLLARY. Suppose $d > 1$ is a divisor of $\frac{p-1}{2}$. Then there exist $(2, 3, d)$ triples (A, B, C) in $PSl_2(p)$.

Next we determine when there are $(2, 3, d)$ triples for divisors of $\frac{p+1}{2}$. Suppose $x \in GF(p^2)^*$ is such that $x^{p+1} = 1$. Then consider the triple of matrices (A, B, C) in $PSU_2(p)$:

$$(2.8). \quad A = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}, \quad B = \begin{bmatrix} \bar{x} \bar{a} & -xb \\ \bar{x} \bar{b} & xa \end{bmatrix}, \quad C = \begin{bmatrix} x & 0 \\ 0 & \bar{x} \end{bmatrix}$$

where $a, b \in GF(p^2)$ satisfy $a\bar{a} + b\bar{b} = 1$.

It is easy to check that $ABC = 1$.

(2.9). LEMMA. Let $d > 2$ be any divisor of $\frac{p+1}{2}$. Then there are $(2, 3, d)$ triples in $PSl_2(p)$.

Proof. Let $x \in GF(p^2)^*$ be any element so that d is the least positive integer satisfying $x^d = \pm 1$. Then the matrix C in (2.8) has order d . Next we choose $a \in GF(p^2)^*$ so that $a(x-x^{-1}) = 1$. Since

$$GF(p) = \{b\bar{b} \mid b \in GF(p^2)\}$$

it follows that there exists $b \in GF(p^2)$ such that $a\bar{a} + b\bar{b} = 1$.

We now prove that the matrices A, B of (2.8) have orders 2, 3 respectively, that is we will show that $a + \bar{a} = 0$ and $ax + \bar{a}\bar{x} = \pm 1$. Since $x^{p+1} = 1$ we have

$$1 = a^p(x-x^{-1})^p = a^p(x^p-x^{-p}) = a^p(x^{-1}-x).$$

This together with $1 = a(x-x^{-1})$ implies that $a^p = -a$, i.e., $a + \bar{a} = 0$. Finally

$$ax + \bar{a}\bar{x} = ax + a^p x^p = ax - ax^{-1} = a(x-x^{-1}) = 1. \quad \text{Q.e.d.}$$

The next theorem proves one half of theorem I of the introduction.

(2.10). THEOREM. Suppose d is a divisor of either $\frac{p-1}{2}$ or $\frac{p+1}{2}$ and suppose $d > 6$. Then there is a $(2, 3, d)$ triple (A, B, C) so that the group generated by A, B, C is $PSl_2(p)$.

Proof. Let (A, B, C) be any $(2, 3, d)$ triple and set $G = \langle A, B, C \rangle =$ the subgroup generated by A, B, C . Since G has elements of order $d > 6$ it

follows that G can not be a subgroup of A_4, S_4, A_5 . Therefore, if $G \neq PSl_2(p)$, it follows that either $G \subseteq D$ or $G \subseteq H$, where D is a maximal dihedral subgroup and H is a maximal solvable subgroup (see (2.2)).

First we assume that $G \subseteq D$. Since B, ABA both have order 3 they must commute, i.e., $(AB)^2 = (BA)^2$. But then we have

$$(AB)^6 = (AB)^2 AB (AB)^2 AB = (BA BA) AB (BA BA) AB = BAB^2 BAB^2 = 1$$

contradicting our hypothesis that $C = (AB)^{-1}$ has order $d > 6$.

Next assume that $G \subseteq H$. Since there is an extension

$$1 \rightarrow \mathbf{Z}_p \rightarrow H \xrightarrow{\theta} \mathbf{Z}_{(p-1)/2} \rightarrow 1$$

we see that $(AB)^6 \in \mathbf{Z}_p$ since A has order 2, B has order 3, and $\theta(A)$ and $\theta(B)$ commute. If $d \mid \frac{p-1}{2}$ then

$$1 = (AB)^{6p} = (AB)^6 \left(\frac{p-1}{2} + \frac{p-1}{2} + 1 \right) = (AB)^6 \quad \text{since} \quad (AB)^{\frac{p-1}{2}} = 1.$$

This contradicts the fact that AB has order $d > 6$. The argument for divisors of $\frac{p+1}{2}$ is similar. Q.e.d.

Summarizing we now know that $PSl_2(p)$ is generated by a $(2, 3, p)$ triple and also by any $(2, 3, d)$ triple, where $d > 6$ and d is a divisor of either $\frac{p-1}{2}$ or $\frac{p+1}{2}$. As far as the problem of minimum genus is concerned it turns out that in addition we only need determine those primes p for which $PSl_2(p)$ is generated by a triple of the form $(3, 3, 4), (2, 5, 5), (2, 4, 5)$.

According to (2.4) a matrix $C \in PSl_2(p)$ has order 4, respectively order 5, if, and only if, $\chi^2 \equiv 2 (p)$, respectively $\chi^2 \pm \chi - 1 \equiv 0 (p)$, where $\chi = \text{trace } C$. But these equations are solvable over \mathbf{Z}_p if, and only if, $p \equiv \pm 1 (8)$, respectively $p \equiv \pm 1 (5)$. Since every element of \mathbf{Z}_p can arise as the trace of some matrix we have $PSl_2(p)$ has elements of order 4, respectively order 5, if, and only if, $p \equiv \pm 1 (8)$, respectively $p \equiv \pm 1 (5)$.

To construct $(3, 3, 4)$ triples consider matrices

$$(2.11). \quad A = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} a & b \\ c & -a+1 \end{bmatrix},$$

$$C = (AB)^{-1} = \begin{bmatrix} 1-a-b & a-1 \\ a-c & c \end{bmatrix}$$

where $-a^2 + a - bc \equiv 1 (p)$.

A and B both have order 3 and C will have order 4 if, and only if, $(1-a-b+c)^2 \equiv 2 \pmod{p}$. Therefore we need to find a, b, c satisfying

$$(2.12). \quad -a^2 + a - bc \equiv 1 \pmod{p} \quad \text{and} \quad (1-a-b+c)^2 \equiv 2 \pmod{p}.$$

Assume $p \equiv \pm 1 \pmod{8}$ so that there is $\alpha \in \mathbf{Z}_p$ with $\alpha^2 \equiv 2 \pmod{p}$. Then (2.12) is equivalent to

$$1 - a - b + c \equiv \alpha \quad \text{and} \quad a^2 - a + bc + 1 \equiv 0$$

which in turn is equivalent to finding b, c so that

$$(2.13). \quad -3 - 4bc \text{ is a quadratic residue mod } p \quad \text{and}$$

$$\frac{1 \pm \sqrt{-3 - 4bc}}{2} \equiv 1 - b + c - \alpha.$$

But this is the same as finding b, c so that

$$(2.14). \quad -3 - 4bc \equiv (1 + 2(-b + c - \alpha))^2.$$

Now solving for c we see that there is a solution, if, and only if, $-3b^2 + (2-4\alpha)b - 3$ is a quadratic residue for some choice of b . But quadratic polynomials always assume at least one quadratic residue and therefore it is possible to satisfy (2.12).

Thus we have proved the following theorem.

(2.15). THEOREM. Suppose $p \equiv \pm 1 \pmod{8}$. Then there are $(3, 3, 4)$ triples in $PSl_2(p)$, one such being given by (2.11), where a, b, c are chosen to satisfy

$$-a^2 + a - bc \equiv 1 \pmod{p} \quad \text{and} \quad (1-a-b+c)^2 \equiv 2 \pmod{p}.$$

We still must prove that $PSl_2(p)$ can be generated by a $(3, 3, 4)$ triple if $p \equiv \pm 1 \pmod{8}$.

(2.16). THEOREM. Suppose $p \equiv \pm 1 \pmod{8}$. Then there are $(3, 3, 4)$ triples in $PSl_2(p)$ and any such triple will generate $PSl_2(p)$.

Proof. Let (A, B, C) be any $(3, 3, 4)$ triple, which exists by (2.15), and let $G = \langle A, B, C \rangle$. We use (2.2) to prove that $G = PSl_2(p)$. First note that none of A_4, S_4, A_5 contain $(3, 3, 4)$ triples. Secondly suppose that $G \subseteq D$, where D is a dihedral group. Since A, B are elements, of odd order (in a dihedral group) they commute and consequently AB will not have order 4.

Finally, suppose $G \subset H$, where H is a maximal solvable subgroup of $PSL_2(p)$. From the existence of the extension $1 \rightarrow \mathbf{Z}_p \rightarrow H \xrightarrow{\theta} \mathbf{Z}_{(p-1)/2} \rightarrow 1$ we see that $AB \in \mathbf{Z}_p$ since $\theta(AB)^4 = 1$ and $\theta(AB)^3 = 1$. But this is impossible since the order of AB is 4. Q.e.d.

To construct $(2, 5, 5)$ or $(2, 4, 5)$ triples in the case $p \equiv 1 \pmod{5}$ consider the matrices

$$(2.17). \quad A = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}, \quad B = \begin{bmatrix} -ax^{-1} & -bx \\ -cx^{-1} & ax \end{bmatrix}, \quad C = \begin{bmatrix} x & 0 \\ 0 & x^{-1} \end{bmatrix}$$

where $a, b, c, x \in GF(p)$ are chosen so that

$$-a^2 - bc = 1, \quad x^5 = 1, \quad x \neq \pm 1.$$

If we also have $p \equiv \pm 1 \pmod{8}$ then we can choose a so that $a^2(x-x^{-1})^2 = 2$, and therefore (A, B, C) will be a $(2, 4, 5)$ triple. On the other hand choosing a so that $\alpha = a(x-x^{-1})$ is a solution of $u^2 \pm u - 1 = 0$ will guarantee that (A, B, C) is a $(2, 5, 5)$ triple.

In the case $p \equiv -1 \pmod{5}$ we think of $PSL_2(p)$ as the projective special unitary group $PSU_2(p)$. Thus we have the matrices

$$(2.18). \quad A = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}, \quad B = \begin{bmatrix} \bar{a} \bar{x} & -bx \\ \bar{b} \bar{x} & ax \end{bmatrix}, \quad C = \begin{bmatrix} x & 0 \\ 0 & \bar{x} \end{bmatrix}$$

where $a, b, x \in GF(p^2)$ are chosen to satisfy

$$a \bar{a} + b \bar{b} = 1, \quad x^5 = 1, \quad x \neq \pm 1.$$

x must also satisfy $x \bar{x} = 1$, that is $x^{p+1} = 1$. Since $p+1 \equiv 0 \pmod{5}$ this follows automatically.

First we choose x so that $x^5 = 1$, $x \neq \pm 1$ and then we choose a so that $a^2(x-x^{-1})^2 = 2$, assuming also that $p \equiv \pm 1 \pmod{8}$. In other words let $\alpha \in GF(p)$ be such that $\alpha^2 = 2$ and then set $a(x-x^{-1}) = \alpha$. But then we have $a(x-x^{-1}) = \alpha = \alpha^p = a^p(x^p-x^{-p}) = a^p(x^{-1}-x) = -\bar{a}(x-x^{-1})$ and hence $a + \bar{a} = 0$. Therefore, with these choices, (2.18) is a $(2, 4, 5)$ triple.

In a similar fashion the matrices in (2.18) will be a $(2, 5, 5)$ triple if $a, b, x \in GF(p^2)$ are chosen to satisfy $a \bar{a} + b \bar{b} = 1$, $x^5 = 1$, $x \neq \pm 1$, $a(x-x^{-1}) = \alpha$, where $\alpha \in GF(p)$ is any solution of $u^2 \pm u - 1 = 0$. As a consequence we have the following result.

(2.19). THEOREM.

(a) If $p \equiv \pm 1 \pmod{5}$ then there are $(2, 5, 5)$ triples in $PSL_2(p)$.

(b) If $p \equiv \pm 1 (5)$ and $p \equiv \pm 1 (8)$ then there are $(2, 4, 5)$ triples in $PSl_2(p)$.

It still remains to prove that we can generate $PSl_2(p)$ by $(2, 5, 5)$ triples or $(2, 4, 5)$ triples.

(2.20). THEOREM. If $p \equiv \pm 1 (5)$ and $p \equiv \pm 1 (8)$ then any $(2, 4, 5)$ triple will generate $PSl_2(p)$.

Proof. Let (A, B, C) be any $(2, 4, 5)$ triple and let $G = \langle A, B, C \rangle$. Because of the orders of A, B, C it readily follows that $G \not\subseteq A_4, S_4, A_5$.

Suppose $G \subseteq D$, where D is a dihedral group of order $p \pm 1$. Then $BC = CB$, since elements of orders > 2 in a dihedral group commute. Therefore $(BC)^4 = C^4$. But also $(BC)^2 = 1$, and this together with $C^5 = 1$ implies that $C = 1$, a contradiction.

Finally suppose $G \subseteq H$, where H is a maximal solvable subgroup. Recall that we have an extension

$$1 \rightarrow \mathbf{Z}_p \rightarrow H \xrightarrow{\theta} \mathbf{Z}_{(p-1)/2} \rightarrow 1.$$

Then $C^4 \in \mathbf{Z}_p$ since $(BC)^2 = 1$ and

$$1 = \theta(BC)^4 = \theta(C^4).$$

From this it follows that the order of C is p , a contradiction. Therefore $G = PSl_2(p)$. Q.e.d.

The generation of $PSl_2(p)$ by $(2, 5, 5)$ triples is more delicate since it is possible to generate A_5 by such triples.

(2.21). THEOREM. If $p \equiv \pm 1 (5)$ then there are $(2, 5, 5)$ triples generating $PSl_2(p)$.

Proof. First we consider the case $p \equiv 1 (5)$. The matrices A, B, C in (2.17) will be a $(2, 5, 5)$ triple if

$$-a^2 - bc = 1, \quad x^5 = 1, \quad x \neq \pm 1, \quad a(x - x^{-1}) = \alpha,$$

where $\alpha \in GF(p)$ is any solution of $u^2 \pm u - 1 = 0$. In particular $\alpha = x + x^{-1}$ is such a solution. In fact $\alpha^2 + \alpha - 1 = 0$.

As before let $G = \langle A, B, C \rangle$. By arguments similar to those of (2.20) we see that $G \not\subseteq A_4, S_4, D$ or H . To show that G can not be a subgroup of A_5 consider the matrix

$$C^2A = \begin{bmatrix} ax^2 & bx^2 \\ cx^{-2} & -ax^2 \end{bmatrix}.$$

The trace of this matrix is

$$\chi = a(x^2 - x^{-2}) = a(x - x^{-1})(x + x^{-1}) = (x + x^{-1})^2.$$

Using (2.4) we can show that C^2A does not have order 2, 3, or 5, and this eliminates A_5 . Hence $G = PSL_2(p)$ in this case.

For the case $p \equiv -1 \pmod{5}$ we choose matrices A, B, C as in (2.18), where now

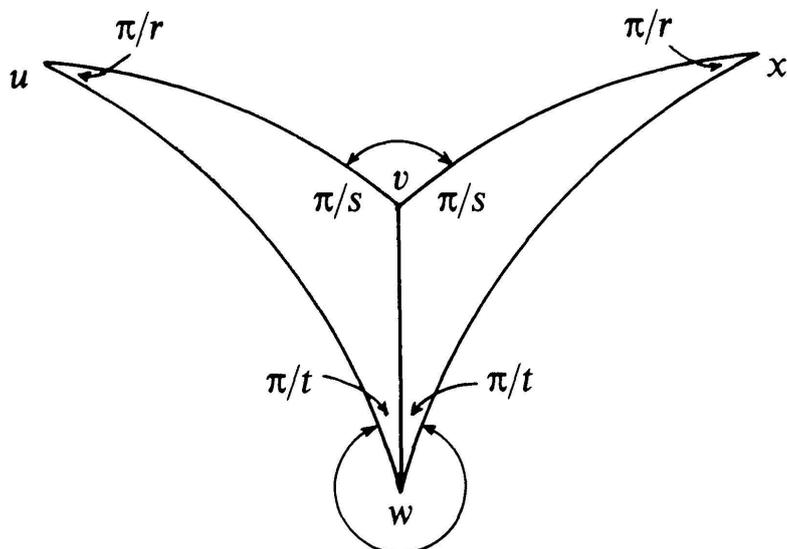
$$a\bar{a} + b\bar{b} = 1, \quad x^5 = 1, \quad x \neq \pm 1, \quad a(x - x^{-1}) = x + x^{-1}.$$

As in the first case we can show that $\langle A, B, C \rangle = PSL_2(p)$. Q.e.d.

Theorems (2.16), (2.20) and (2.21) now establish half of theorem II in the introduction. The other half follows from the result below.

(2.22). THEOREM. Suppose G is a finite group and (A, B, C) is an (r, s, t) triple generating G . If $1 \rightarrow \Delta \rightarrow T(r, s, t) \rightarrow G \rightarrow 1$ is the associated extension then the genus of H/Δ is $1 + \frac{|G|}{2} \left(1 - \frac{1}{r} - \frac{1}{s} - \frac{1}{t} \right)$.

Proof. A fundamental domain for the action of $T(r, s, t)$ on P , where P is the appropriate plane, consists of two copies of a triangle whose angles are $\pi/r, \pi/s, \pi/t$ (see the diagram)



A, B, C are rotations about u, v, w through angles $2\pi/r, 2\pi/s, 2\pi/t$.

The only identifications under the action are: vu gets identified to vx and wu gets identified to wx . It follows that $P/T(r, s, t)$ is the 2 sphere and the branched covering $P/\Delta \rightarrow P/T(r, s, t)$ has 3 branch points coming from the vertices u, v, w .

Now notice that Δ is torsion free. This follows from the facts:

(1) the elements of finite order in $T(r, s, t)$ are the conjugates of A, B, C .

(2) elements of finite order in $T(r, s, t)$ map to elements of the same order in G . From this it follows that the orders of the branch points are r, s, t respectively.

Finally we consider the Riemann-Hurwitz formula:

$$\chi(P/\Delta) = |G| \left(\chi(P/T(r, s, t)) - \left(1 - \frac{1}{r}\right) - \left(1 - \frac{1}{s}\right) - \left(1 - \frac{1}{t}\right) \right)$$

i.e.,
$$2 - 2g = |G| \left(\frac{1}{r} + \frac{1}{s} + \frac{1}{t} - 1 \right).$$

Therefore
$$g = 1 + \frac{|G|}{2} \left(1 - \frac{1}{r} - \frac{1}{s} - \frac{1}{t} \right) \quad \text{Q.e.d.}$$

§ 3. CONFORMAL ACTIONS ON SURFACES OF LEAST GENUS

If (A, B, C) is an (r, s, t) triple generating $PSl_2(p)$ then we have a short exact sequence

$$1 \rightarrow \Delta \rightarrow T(r, s, t) \rightarrow PSl_2(p) \rightarrow 1$$

where Δ is torsion free. Then it follows that $H/T(r, s, t)$ is S^2 and the branched covering $H/\Delta \rightarrow H/T(r, s, t)$ has 3 branch points with orders r, s, t .

Conversely we have:

(3.1). THEOREM. *If S is a Riemann surface of least genus for $PSl_2(p)$ then $S/PSl_2(p)$ is S^2 and $\pi: S \rightarrow S/PSl_2(p)$ has 3 branch points.*

Proof. There exists a short exact sequence $1 \rightarrow \Delta \rightarrow T(2, 3, p) \rightarrow PSl_2(p) \rightarrow 1$ arising from a $(2, 3, p)$ triple and consequently

$$\text{genus}(H/\Delta) = 1 + \frac{|G|}{2} \left(\frac{1}{6} - \frac{1}{p} \right).$$