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REPRESENTING $PSL_2(p)$ ON A RIEMANN SURFACE OF LEAST GENUS

by Henry GLOVER and Denis SJERVE ¹⁾

§ 1. INTRODUCTION

Given any finite group G there exists a closed Riemann surface S and an effective action $G \times S \rightarrow S$ by conformal automorphisms (here conformal means analytic). Therefore it makes sense to ask what is the least genus of such surfaces S . Recall that when the answer is that the genus equals zero (i.e. G acts on the two sphere) then G is from the list \mathbf{Z}/n , D_n , A_4 , S_4 or A_5 . The purpose of this paper is to determine this minimum genus for the simple groups $PSL_2(p)$, where $p \geq 5$ is a prime. Since given any finite group G and Riemann surface T there exists a regular branched covering $p: S \rightarrow T$ such that i) G is the group of branched covering transformations of p (i.e. $T = S/G$) and ii) G is the full group of automorphisms of S [Gr], it seems most interesting to realize G as the full group of automorphisms of a Riemann surface of least genus. In a sequel to this paper [GS] we will prove that this always happens when $p \not\equiv \pm 1 \pmod{8}$ or $\pmod{5}$ but *may* fail for these congruence equalities. When it does fail $PSL_2(p)$ will have index two in the full group of automorphisms. In addition, a particularly simple situation occurs when $p: S \rightarrow S/G$ has exactly three branch points. Our results always give this for $PSL_2(p)$. We conjecture analogous results for every finite simple group and we seek to relate these ideas to "moonshine" for simple groups [FLM]. In order to state our results we need some notation:

- (1) $PSL_2(p^k)$ is the projective special linear group of 2×2 matrices over the Galois field $GF(p^k)$.
- (2) $\Gamma = PSL_2(\mathbf{Z})$ is the classical modular group. Geometrically Γ is just the group of integral linear fractional transformations of the upper half plane H , that is transformations of the form $z \rightarrow \frac{az + b}{cz + d}$, where a, b, c, d

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are integers so that $ad - bc = 1$. Algebraically Γ is the unimodular group $Sl_2(\mathbf{Z})$ modulo its center $= \{\pm I\}$.

A result of Newman [N] is that mod p reduction of entries gives an epimorphism $\Gamma \twoheadrightarrow PSl_2(p)$, and therefore an exact sequence $1 \rightarrow \Delta \rightarrow \Gamma \rightarrow PSl_2(p) \rightarrow 1$. Now Δ is a Fuchsian group and therefore $PSl_2(p)$ is acting conformally on the open Riemann surface H/Δ . By adding parabolic points we obtain a closed Riemann surface $\overline{H/\Delta}$ and a conformal action on $\overline{H/\Delta}$ by extension. According to [G] the genus of $\overline{H/\Delta}$ is

$$1 + \frac{|PSl_2(p)|}{2} \left(\frac{1}{6} - \frac{1}{p} \right) = 1 + \frac{p(p^2-1)}{4} \left(\frac{1}{6} - \frac{1}{p} \right)$$

where $|PSl_2(p)| = \frac{p(p^2-1)}{2}$ is the order of $PSl_2(p)$.

Definition. For any finite group G we let *genus* (G) denote the least genus of all Riemann surfaces S for which there exists an effective conformal action $G \times S \rightarrow S$. We note that *genus* (G) has also been called the symmetric genus of G in the literature.

Thus we certainly have $\text{genus}(PSl_2(p)) \leq 1 + \frac{p(p^2-1)}{4} \left(\frac{1}{6} - \frac{1}{p} \right)$. Putting $p = 5$ then gives $\text{genus}(PSl_2(5)) = 0$, and therefore we will tacitly assume in all that follows that $p \geq 7$.

For $p = 7, 11$ we get the inequalities $\text{genus}(PSl_2(7)) \leq 3$ and $\text{genus}(PSl_2(11)) \leq 26$. It will turn out that these inequalities are equalities (see the corollary of the introduction). The action of $PSl_2(7)$ on a surface of genus 3 is the action of the simple group of order 168 considered by Klein.

This inequality strongly suggests that $\text{genus}(PSl_2(p))$ can be calculated by realizing $PSl_2(p)$ as an epimorphic image of Γ , or some other Fuchsian group, and then minimizing over all such epimorphisms. For example Γ has the presentation:

$$\Gamma = \{S, T \mid S^2 = (ST)^3 = 1\},$$

where $S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

Reducing coefficients mod p leads to a presentation of $PSl_2(p)$, namely

$$PSl_2(p) = \{A, B, C \mid A^2 = B^3 = C^p = ABC = 1, \text{ETC}\}$$

where we have made the substitutions $A = S$, $B = ST$ and $C = T^{-1}$. We have written the presentation in this manner so that it becomes clear that $PSl_2(p)$ is an epimorphic image of the triangle group

$$T(2, 3, p) = \{A, B, C \mid A^2 = B^3 = C^p = ABC = 1\}.$$

Recall that if r, s, t are integers ≥ 2 then $T(r, s, t)$ is the group of orientation preserving symmetries of the appropriate plane generated by rotations of $2\pi/r$, $2\pi/s$ and $2\pi/t$, respectively, about the vertices of a triangle having angles π/r , π/s and π/t respectively. The plane is spherical if $1/r + 1/s + 1/t > 1$, euclidean if $1/r + 1/s + 1/t = 1$, and hyperbolic if $1/r + 1/s + 1/t < 1$. See Magnus [M] for more details.

Using the above presentation of $PSl_2(p)$ leads to an exact sequence $1 \rightarrow \Delta \rightarrow T(2, 3, p) \rightarrow PSl_2(p) \rightarrow 1$ and an effective conformal action of $PSl_2(p)$ on the closed Riemann surface H/Δ . Again we have

$$\text{genus}(H/\Delta) = 1 + \frac{p(p^2-1)}{4} \left(\frac{1}{6} - \frac{1}{p} \right)$$

so there is no improvement. But now the idea is clear: find all triples (r, s, t) for which there is an exact sequence $1 \rightarrow \Delta \rightarrow T(r, s, t) \rightarrow PSl_2(p) \rightarrow 1$, compute the genus of H/Δ for any such extension, and then minimize over all possible triples. It turns out that this procedure gives genus $(PSl_2(p))$ because more branch points always gives a higher genus.

If $p \geq 13$ we make the definition $d = \min\{e \mid e \geq 7 \text{ and either } e \mid \frac{p-1}{2} \text{ or } e \mid \frac{p+1}{2}\}$. Then our results are:

THEOREM I. Assume $p \geq 13$. Then there exists a short exact sequence $1 \rightarrow \Delta \rightarrow T(2, 3, d) \rightarrow PSl_2(p) \rightarrow 1$ and the genus of H/Δ is

$$1 + \frac{p(p^2-1)}{4} \left(\frac{1}{6} - \frac{1}{d} \right).$$

THEOREM II.

- (a) If $p \equiv \pm 1 (5)$ then there exists a short exact sequence $1 \rightarrow \Delta \rightarrow T(2, 5, 5) \rightarrow PSl_2(p) \rightarrow 1$ and the genus of H/Δ is $1 + \frac{p(p^2-1)}{40}$.
- (b) If $p \equiv \pm 1 (8)$ then there exists a short exact sequence $1 \rightarrow \Delta \rightarrow T(3, 3, 4) \rightarrow PSl_2(p) \rightarrow 1$ and the genus of H/Δ is $1 + \frac{p(p^2-1)}{48}$.

- (c) If $p \equiv \pm 1 (5)$ and $p \equiv \pm 1 (8)$ then there exists a short exact sequence $1 \rightarrow \Delta \rightarrow T(2, 4, 5) \rightarrow PSl_2(p) \rightarrow 1$ and the genus of H/Δ is $1 + \frac{p(p^2-1)}{80}$.

Then we will prove that genus $(PSl_2(p))$ is obtained by minimizing over all the possibilities above.

The result of this minimization is

COROLLARY. The genus of $PSl_2(p)$ is given as follows:

- (a) $g = 1 + \frac{p(p^2-1)}{4} \left(\frac{1}{6} - \frac{1}{p} \right)$ if $p = 5, 7, 11$,
- (b) $g = 1 + \frac{p(p^2-1)}{40}$ if $p \geq 13$, $p \equiv \pm 1 (5)$, $p \not\equiv \pm 1 (8)$
and $d \geq 15$,
- (c) $g = 1 + \frac{p(p^2-1)}{48}$ if $p \geq 13$, $p \not\equiv \pm 1 (5)$, $p \equiv \pm 1 (8)$
and $d \geq 12$,
- (d) $g = 1 + \frac{p(p^2-1)}{80}$ if $p \geq 13$, $p \equiv \pm 1 (5)$, $p \equiv \pm 1 (8)$
and $d \geq 9$,
- (e) $g = 1 + \frac{p(p^2-1)}{4} \left(\frac{1}{6} - \frac{1}{d} \right)$ in all other cases.

In fact the least genus g always comes from the branched covering space action on the Riemann surface $S = H/\Delta$ associated to some extension $1 \rightarrow \Delta \rightarrow T(r, s, t) \rightarrow PSl_2(p) \rightarrow 1$, where

$$(r, s, t) = \begin{cases} (2, 3, p) & \text{if } p = 5, 7, 11, \\ (2, 5, 5) & \text{if } p \geq 13, p \equiv \pm 1 (5), p \not\equiv \pm 1 (8) \text{ and } d \geq 15, \\ (3, 3, 4) & \text{if } p \geq 13, p \not\equiv \pm 1 (5), p \equiv \pm 1 (8) \text{ and } d \geq 12, \\ (2, 4, 5) & \text{if } p \geq 13, p \equiv \pm 1 (5), p \equiv \pm 1 (8) \text{ and } d \geq 9, \\ (2, 3, d) & \text{in all other cases.} \end{cases}$$

It turns out that other triples (r, s, t) are not relevant for the determination of the minimal genus.

In most cases the answer is $(r, s, t) = (2, 3, d)$. For $p \leq 617$ the triple $(2, 5, 5)$ occurs once exactly, namely for $p = 509$, $(3, 3, 4)$ occurs exactly three

times, namely for $p = 103, 137$ and 569 and $(2, 4, 5)$ occurs exactly six times, for $p = 199, 239, 359, 439, 521$ and 599 .

If $S = H/\Delta$ is the surface of minimal genus for $PSl_2(p)$ coming from one of the extensions above then the orbit manifold $S/PSl_2(p)$ is the 2-sphere S^2 and the quotient map $S \rightarrow S^2$ is a branched covering with exactly 3 branch points. One of the most important steps in the proof of the main result of this paper is the converse, namely if S is a Riemann surface of least genus for the group $G = PSl_2(p)$ then $S/G = S^2$ and $S \rightarrow S^2$ is a branched covering with exactly 3 branch points (see section 3). Note that a related notion of genus, "the Cayley genus of a group" has been studied by others, among them Tucker [T]. Earlier results can be found in Hurwitz [H] and Burnside [B].

The remainder of this paper is organized as follows. In section 2 we describe various ways of generating $PSl_2(p)$ and then prove theorems I and II. Section 3 proves that if S is a Riemann surface of least genus for $PSl_2(p)$ then $S/PSl_2(p)$ is a 2-sphere S^2 and the branched covering $S \rightarrow S^2$ has exactly 3 branch points. The calculation of genus ($PSl_2(p)$) then follows from the results of section 2.

Finally we would like to thank Bomshik Chang for help with the group theory of $PSl_2(p)$. The first author would like to thank the University of British Columbia for its hospitality to him during the time this research was done.

§ 2. GENERATING TRIPLES FOR $PSl_2(p)$

Our goal in this section is to find triples (r, s, t) for which there are epimorphisms $T(r, s, t) \twoheadrightarrow PSl_2(p)$. In other words, given integers $r, s, t \geq 2$ are there matrices $A, B, C \in PSl_2(p)$ so that A, B, C generate $PSl_2(p)$ and $A^r = B^s = C^t = ABC = 1$? Throughout this section a standard reference for the group theory is Suzuki [S].

The spherical triangle groups are given in the following table

TABLE I

<i>triple</i>	<i>triangle group</i>	<i>order</i>
$(2, 2, n)$	dihedral	$2n$
$(2, 3, 3)$	tetrahedral (A_4)	12
$(2, 3, 4)$	octahedral (S_4)	24
$(2, 3, 5)$	icosahedral (A_5)	60