L'Enseignement Mathématique, t. 31 (1985), p. 305-325

REPRESENTING $PSl_2(p)$ ON A RIEMANN SURFACE OF LEAST GENUS

by Henry GLOVER and Denis SJERVE [1])

§ 1. INTRODUCTION

Given any finite group G there exists a closed Riemann surface S and an effective action $G \times S \to S$ by conformal automorphisms (here conformal means analytic). Therefore it makes sense to ask what is the least genus of such surfaces S. Recall that when the answer is that the genus equals zero (i.e. G acts on the two sphere) then G is from the list $\mathbf{Z}/n$, D_n, A_4, S_4 or A_5. The purpose of this paper is to determine this minimum genus for the simple groups $PSl_2(p)$, where $p \geqslant 5$ is a prime. Since given any finite group G and Riemann surface T there exists a regular branched covering $p: S \to T$ such that i) G is the group of branched covering transformations of p (i.e. $T = S/G$) and ii) G is the full group of automorphisms of S [Gr], it seems most interesting to realize G as the full group of automorphisms of a Riemann surface of least genus. In a sequel to this paper [GS] we will prove that this always happens when $p \not\equiv \pm 1 \bmod 8$ or mod 5 but *may* fail for these congruence equalities. When it does fail $PSl_2(p)$ will have index two in the full group of automorphisms. In addition, a particularly simple situation occurs when $p: S \to S/G$ has exactly three branch points. Our results always give this for $PSl_2(p)$. We conjecture analogus results for every finite simple group and we seek to relate these ideas to "moonshine" for simple groups [FLM]. In order to state our results we need some notation:

(1) $PSl_2(p^k)$ is the projective special linear group of 2×2 matrices over the Galois field $GF(p^k)$.

(2) $\Gamma = PSl_2(\mathbf{Z})$ is the classical modular group. Geometrically Γ is just the group of integral linear fractional transformations of the upper half plane H, that is transformations of the form $z \to \dfrac{az + b}{cz + d}$, where a, b, c, d

[1]) Research partially supported by N.S.E.R.C. grant 67-7218.

are integers so that $ad - bc = 1$. Algebraically Γ is the unimodular group $Sl_2(\mathbf{Z})$ modulo its center $= \{\pm I\}$.

A result of Newman [N] is that mod p reduction of entries gives an epimorphism $\Gamma \twoheadrightarrow PSl_2(p)$, and therefore an exact sequence $1 \to \Delta \to \Gamma \to PSl_2(p) \to 1$. Now Δ is a Fuchsian group and therefore $PSl_2(p)$ is acting conformally on the open Riemann surface H/Δ. By adding parabolic points we obtain a closed Riemann surface $\overline{H/\Delta}$ and a conformal action on $\overline{H/\Delta}$ by extension. According to [G] the genus of $\overline{H/\Delta}$ is

$$1 + \frac{|PSl_2(p)|}{2}\left(\frac{1}{6} - \frac{1}{p}\right) = 1 + \frac{p(p^2-1)}{4}\left(\frac{1}{6} - \frac{1}{p}\right)$$

where $|PSl_2(p)| = \dfrac{p(p^2-1)}{2}$ is the order of $PSl_2(p)$.

Definition. For any finite group G we let *genus* (G) denote the least genus of all Riemann surfaces S for which there exists an effective conformal action $G \times S \to S$. We note that genus (G) has also been called the symmetric genus of G in the literature.

Thus we certainly have genus $\big(PSl_2(p)\big) \leqslant 1 + \dfrac{p(p^2-1)}{4}\left(\dfrac{1}{6} - \dfrac{1}{p}\right)$. Putting $p = 5$ then gives genus $\big(PSl_2(5)\big) = 0$, and therefore we will tacitly assume in all that follows that $p \geqslant 7$.

For $p = 7, 11$ we get the inequalities genus $\big(PSl_2(7)\big) \leqslant 3$ and genus $\big(PSl_2(11)\big) \leqslant 26$. It will turn out that these inequalities are equalities (see the corollary of the introduction). The action of $PSl_2(7)$ on a surface of genus 3 is the action of the simple group of order 168 considered by Klein.

This inequality strongly suggests that genus $\big(PSl_2(p)\big)$ can be calculated by realizing $PSl_2(p)$ as an epimorphic image of Γ, or some other Fuchsian group, and then minimizing over all such epimorphisms. For example Γ has the presentation:

$$\Gamma = \{S, T \mid S^2 = (ST)^3 = 1\},$$

$$\text{where} \quad S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Reducing coefficients mod p leads to a presentation of $PSl_2(p)$, namely

$$PSl_2(p) = \{A, B, C \mid A^2 = B^3 = C^p = ABC = 1, ETC\}$$

where we have made the substitutions $A = S$, $B = ST$ and $C = T^{-1}$. We have written the presentation in this manner so that it becomes clear that $PSl_2(p)$ is an epimorphic image of the triangle group

$$T(2, 3, p) = \{A, B, C \mid A^2 = B^3 = C^p = ABC = 1\}.$$

Recall that if r, s, t are integers $\geqslant 2$ then $T(r, s, t)$ is the group of orientation preserving symmetries of the appropriate plane generated by rotations of $2\pi/r$, $2\pi/s$ and $2\pi/t$, respectively, about the vertices of a triangle having angles π/r, π/s and π/t respectively. The plane is spherical if $1/r + 1/s + 1/t > 1$, euclidean if $1/r + 1/s + 1/t = 1$, and hyperbolic if $1/r + 1/s + 1/t < 1$. See Magnus [M] for more details.

Using the above presentation of $PSl_2(p)$ leads to an exact sequence $1 \to \Delta \to T(2, 3, p) \to PSl_2(p) \to 1$ and an effective conformal action of $PSl_2(p)$ on the closed Riemann surface H/Δ. Again we have

$$\text{genus } (H/\Delta) = 1 + \frac{p(p^2-1)}{4}\left(\frac{1}{6} - \frac{1}{p}\right)$$

so there is no improvement. But now the idea is clear: find all triples (r, s, t) for which there is an exact sequence $1 \to \Delta \to T(r, s, t) \to PSl_2(p) \to 1$, compute the genus of H/Δ for any such extension, and then minimize over all possible triples. It turns out that this procedure gives genus $(PSl_2(p))$ because more branch points always gives a higher genus.

If $p \geqslant 13$ we make the definition $d = \min\{e \mid e \geqslant 7$ and either $e \mid \frac{p-1}{2}$ or $e \mid \frac{p+1}{2}\}$. Then our results are:

THEOREM I. *Assume* $p \geqslant 13$. *Then there exists a short exact sequence* $1 \to \Delta \to T(2, 3, d) \to PSl_2(p) \to 1$ *and the genus of* H/Δ *is*

$$1 + \frac{p(p^2-1)}{4}\left(\frac{1}{6} - \frac{1}{d}\right).$$

THEOREM II.

(a) *If* $p \equiv \pm 1\,(5)$ *then there exists a short exact sequence* $1 \to \Delta \to T(2, 5, 5) \to PSl_2(p) \to 1$ *and the genus of* H/Δ *is* $1 + \dfrac{p(p^2-1)}{40}$.

(b) *If* $p \equiv \pm 1\,(8)$ *then there exists a short exact sequence* $1 \to \Delta \to T(3, 3, 4) \to PSl_2(p) \to 1$ *and the genus of* H/Δ *is* $1 + \dfrac{p(p^2-1)}{48}$.

(c) *If* $p \equiv \pm 1\,(5)$ *and* $p \equiv \pm 1\,(8)$ *then there exists a short exact sequence* $1 \to \Delta \to T(2, 4, 5) \to PSl_2(p) \to 1$ *and the genus of* H/Δ *is* $1 + \dfrac{p(p^2-1)}{80}$.

Then we will prove that genus $(PSl_2(p))$ is obtained by minimizing over all the possibilities above.

The result of this minimization is

COROLLARY. *The genus of* $PSl_2(p)$ *is given as follows:*

(a) $g = 1 + \dfrac{p(p^2-1)}{4}\left(\dfrac{1}{6} - \dfrac{1}{p}\right)$ *if* $p = 5, 7, 11$,

(b) $g = 1 + \dfrac{p(p^2-1)}{40}$ *if* $p \geqslant 13$, $p \equiv \pm 1\,(5)$, $p \not\equiv \pm 1\,(8)$ *and* $d \geqslant 15$,

(c) $g = 1 + \dfrac{p(p^2-1)}{48}$ *if* $p \geqslant 13$, $p \not\equiv \pm 1\,(5)$, $p \equiv \pm 1\,(8)$ *and* $d \geqslant 12$,

(d) $g = 1 + \dfrac{p(p^2-1)}{80}$ *if* $p \geqslant 13$, $p \equiv \pm 1\,(5)$, $p \equiv \pm 1\,(8)$ *and* $d \geqslant 9$,

(e) $g = 1 + \dfrac{p(p^2-1)}{4}\left(\dfrac{1}{6} - \dfrac{1}{d}\right)$ *in all other cases.*

In fact the least genus g always comes from the branched covering space action on the Riemann surface $S = H/\Delta$ associated to some extension $1 \to \Delta \to T(r, s, t) \to PSl_2(p) \to 1$,where

$$(r, s, t) = \begin{cases} (2, 3, p) & \text{if} \quad p = 5, 7, 11, \\ (2, 5, 5) & \text{if} \quad p \geqslant 13, \quad p \equiv \pm 1\,(5), \quad p \not\equiv \pm 1\,(8) \quad \text{and} \quad d \geqslant 15, \\ (3, 3, 4) & \text{if} \quad p \geqslant 13, \quad p \not\equiv \pm 1\,(5), \quad p \equiv \pm 1\,(8) \quad \text{and} \quad d \geqslant 12, \\ (2, 4, 5) & \text{if} \quad p \geqslant 13, \quad p \equiv \pm 1\,(5), \quad p \equiv \pm 1\,(8) \quad \text{and} \quad d \geqslant 9, \\ (2, 3, d) & \text{in all other cases.} \end{cases}$$

It turns out that other triples (r, s, t) are not relevant for the determination of the minimal genus.

In most cases the answer is $(r, s, t) = (2, 3, d)$. For $p \leqslant 617$ the triple $(2, 5, 5)$ occurs once exactly, namely for $p = 509$, $(3, 3, 4)$ occurs exactly three times, namely for $p = 103$, 137 and 569 and (2, 4, 5) occurs exactly six times, for $p = 199$, 239, 359, 439, 521 and 599.

If $S = H/\Delta$ is the surface of minimal genus for $PSl_2(p)$ coming from one of the extensions above then the orbit manifold $S/PSl_2(p)$ is the 2-sphere S^2 and the quotient map $S \to S^2$ is a branched covering with exactly 3 branch points. One of the most important steps in the proof of the main result of this paper is the converse, namely if S is a Riemann surface of least genus for the group $G = PSl_2(p)$ then $S/G = S^2$ and $S \to S^2$ is a branched covering with exactly 3 branch points (see section 3). Note that a related notion of genus, "the Cayley genus of a group" has been studied by others, among them Tucker [T]. Earlier results can be found in Hurwitz [H] and Burnside [B].

The remainder of this paper is organized as follows. In section 2 we describe various ways of generating $PSl_2(p)$ and then prove theorems I and II. Section 3 proves that if S is a Riemann surface of least genus for $PSl_2(p)$ then $S/PSl_2(p)$ is a 2-sphere S^2 and the branched covering $S \to S^2$ has exactly 3 branch points. The calculation of genus $(PSl_2(p))$ then follows from the results of section 2.

Finally we would like to thank Bomshik Chang for help with the group theory of $PSl_2(p)$. The first author would like to thank the University of British Columbia for its hospitality to him during the time this research was done.

§ 2. Generating Triples for $PSl_2(p)$

Our goal in this section is to find triples (r, s, t) for which there are epimorphisms $T(r, s, t) \twoheadrightarrow PSl_2(p)$. In other words, given integers $r, s, t \geqslant 2$ are there matrices $A, B, C \in PSl_2(p)$ so that A, B, C generate $PSl_2(p)$ and $A^r = B^s = C^t = ABC = 1$? Throughout this section a standard reference for the group theory is Suzuki [S].

The spherical triangle groups are given in the following table

Table I

triple	*triangle group*	*order*
$(2, 2, n)$	dihedral	$2n$
$(2, 3, 3)$	tetrahedral (A_4)	12
$(2, 3, 4)$	octahedral (S_4)	24
$(2, 3, 5)$	icosahedral (A_5)	60