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## REPRESENTING $PSl_2(p)$ ON A RIEMANN SURFACE OF LEAST GENUS

by Henry GLOVER and Denis SJERVE <sup>1)</sup>

### § 1. INTRODUCTION

Given any finite group  $G$  there exists a closed Riemann surface  $S$  and an effective action  $G \times S \rightarrow S$  by conformal automorphisms (here conformal means analytic). Therefore it makes sense to ask what is the least genus of such surfaces  $S$ . Recall that when the answer is that the genus equals zero (i.e.  $G$  acts on the two sphere) then  $G$  is from the list  $\mathbf{Z}/n$ ,  $D_n$ ,  $A_4$ ,  $S_4$  or  $A_5$ . The purpose of this paper is to determine this minimum genus for the simple groups  $PSl_2(p)$ , where  $p \geq 5$  is a prime. Since given any finite group  $G$  and Riemann surface  $T$  there exists a regular branched covering  $p: S \rightarrow T$  such that i)  $G$  is the group of branched covering transformations of  $p$  (i.e.  $T=S/G$ ) and ii)  $G$  is the full group of automorphisms of  $S$  [Gr], it seems most interesting to realize  $G$  as the full group of automorphisms of a Riemann surface of least genus. In a sequel to this paper [GS] we will prove that this always happens when  $p \not\equiv \pm 1 \pmod{8}$  or  $\pmod{5}$  but *may* fail for these congruence equalities. When it does fail  $PSl_2(p)$  will have index two in the full group of automorphisms. In addition, a particularly simple situation occurs when  $p: S \rightarrow S/G$  has exactly three branch points. Our results always give this for  $PSl_2(p)$ . We conjecture analogous results for every finite simple group and we seek to relate these ideas to "moonshine" for simple groups [FLM]. In order to state our results we need some notation:

- (1)  $PSl_2(p^k)$  is the projective special linear group of  $2 \times 2$  matrices over the Galois field  $GF(p^k)$ .
- (2)  $\Gamma = PSl_2(\mathbf{Z})$  is the classical modular group. Geometrically  $\Gamma$  is just the group of integral linear fractional transformations of the upper half plane  $H$ , that is transformations of the form  $z \rightarrow \frac{az + b}{cz + d}$ , where  $a, b, c, d$

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are integers so that  $ad - bc = 1$ . Algebraically  $\Gamma$  is the unimodular group  $Sl_2(\mathbf{Z})$  modulo its center  $= \{\pm I\}$ .

A result of Newman [N] is that mod  $p$  reduction of entries gives an epimorphism  $\Gamma \rightarrow PSl_2(p)$ , and therefore an exact sequence  $1 \rightarrow \Delta \rightarrow \Gamma \rightarrow PSl_2(p) \rightarrow 1$ . Now  $\Delta$  is a Fuchsian group and therefore  $PSl_2(p)$  is acting conformally on the open Riemann surface  $H/\Delta$ . By adding parabolic points we obtain a closed Riemann surface  $\overline{H/\Delta}$  and a conformal action on  $\overline{H/\Delta}$  by extension. According to [G] the genus of  $\overline{H/\Delta}$  is

$$1 + \frac{|PSl_2(p)|}{2} \left( \frac{1}{6} - \frac{1}{p} \right) = 1 + \frac{p(p^2-1)}{4} \left( \frac{1}{6} - \frac{1}{p} \right)$$

where  $|PSl_2(p)| = \frac{p(p^2-1)}{2}$  is the order of  $PSl_2(p)$ .

*Definition.* For any finite group  $G$  we let *genus* ( $G$ ) denote the least genus of all Riemann surfaces  $S$  for which there exists an effective conformal action  $G \times S \rightarrow S$ . We note that *genus* ( $G$ ) has also been called the symmetric genus of  $G$  in the literature.

Thus we certainly have  $\text{genus}(PSl_2(p)) \leq 1 + \frac{p(p^2-1)}{4} \left( \frac{1}{6} - \frac{1}{p} \right)$ . Putting  $p = 5$  then gives  $\text{genus}(PSl_2(5)) = 0$ , and therefore we will tacitly assume in all that follows that  $p \geq 7$ .

For  $p = 7, 11$  we get the inequalities  $\text{genus}(PSl_2(7)) \leq 3$  and  $\text{genus}(PSl_2(11)) \leq 26$ . It will turn out that these inequalities are equalities (see the corollary of the introduction). The action of  $PSl_2(7)$  on a surface of genus 3 is the action of the simple group of order 168 considered by Klein.

This inequality strongly suggests that  $\text{genus}(PSl_2(p))$  can be calculated by realizing  $PSl_2(p)$  as an epimorphic image of  $\Gamma$ , or some other Fuchsian group, and then minimizing over all such epimorphisms. For example  $\Gamma$  has the presentation:

$$\Gamma = \{S, T \mid S^2 = (ST)^3 = 1\},$$

where  $S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$  and  $T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ .

Reducing coefficients mod  $p$  leads to a presentation of  $PSl_2(p)$ , namely

$$PSl_2(p) = \{A, B, C \mid A^2 = B^3 = C^p = ABC = 1, \text{ETC}\}$$

where we have made the substitutions  $A = S$ ,  $B = ST$  and  $C = T^{-1}$ . We have written the presentation in this manner so that it becomes clear that  $PSl_2(p)$  is an epimorphic image of the triangle group

$$T(2, 3, p) = \{A, B, C \mid A^2 = B^3 = C^p = ABC = 1\}.$$

Recall that if  $r, s, t$  are integers  $\geq 2$  then  $T(r, s, t)$  is the group of orientation preserving symmetries of the appropriate plane generated by rotations of  $2\pi/r$ ,  $2\pi/s$  and  $2\pi/t$ , respectively, about the vertices of a triangle having angles  $\pi/r$ ,  $\pi/s$  and  $\pi/t$  respectively. The plane is spherical if  $1/r + 1/s + 1/t > 1$ , euclidean if  $1/r + 1/s + 1/t = 1$ , and hyperbolic if  $1/r + 1/s + 1/t < 1$ . See Magnus [M] for more details.

Using the above presentation of  $PSl_2(p)$  leads to an exact sequence  $1 \rightarrow \Delta \rightarrow T(2, 3, p) \rightarrow PSl_2(p) \rightarrow 1$  and an effective conformal action of  $PSl_2(p)$  on the closed Riemann surface  $H/\Delta$ . Again we have

$$\text{genus}(H/\Delta) = 1 + \frac{p(p^2-1)}{4} \left( \frac{1}{6} - \frac{1}{p} \right)$$

so there is no improvement. But now the idea is clear: find all triples  $(r, s, t)$  for which there is an exact sequence  $1 \rightarrow \Delta \rightarrow T(r, s, t) \rightarrow PSl_2(p) \rightarrow 1$ , compute the genus of  $H/\Delta$  for any such extension, and then minimize over all possible triples. It turns out that this procedure gives genus  $(PSl_2(p))$  because more branch points always gives a higher genus.

If  $p \geq 13$  we make the definition  $d = \min\{e \mid e \geq 7 \text{ and either } e \mid \frac{p-1}{2} \text{ or } e \mid \frac{p+1}{2}\}$ . Then our results are:

**THEOREM I.** *Assume  $p \geq 13$ . Then there exists a short exact sequence  $1 \rightarrow \Delta \rightarrow T(2, 3, d) \rightarrow PSl_2(p) \rightarrow 1$  and the genus of  $H/\Delta$  is*

$$1 + \frac{p(p^2-1)}{4} \left( \frac{1}{6} - \frac{1}{d} \right).$$

**THEOREM II.**

- (a) *If  $p \equiv \pm 1 \pmod{5}$  then there exists a short exact sequence  $1 \rightarrow \Delta \rightarrow T(2, 5, 5) \rightarrow PSl_2(p) \rightarrow 1$  and the genus of  $H/\Delta$  is  $1 + \frac{p(p^2-1)}{40}$ .*
- (b) *If  $p \equiv \pm 1 \pmod{8}$  then there exists a short exact sequence  $1 \rightarrow \Delta \rightarrow T(3, 3, 4) \rightarrow PSl_2(p) \rightarrow 1$  and the genus of  $H/\Delta$  is  $1 + \frac{p(p^2-1)}{48}$ .*

- (c) If  $p \equiv \pm 1 (5)$  and  $p \equiv \pm 1 (8)$  then there exists a short exact sequence  $1 \rightarrow \Delta \rightarrow T(2, 4, 5) \rightarrow PSl_2(p) \rightarrow 1$  and the genus of  $H/\Delta$  is  $1 + \frac{p(p^2-1)}{80}$ .

Then we will prove that genus ( $PSl_2(p)$ ) is obtained by minimizing over all the possibilities above.

The result of this minimization is

COROLLARY. The genus of  $PSl_2(p)$  is given as follows:

- (a)  $g = 1 + \frac{p(p^2-1)}{4} \left( \frac{1}{6} - \frac{1}{p} \right)$  if  $p = 5, 7, 11$ ,
- (b)  $g = 1 + \frac{p(p^2-1)}{40}$  if  $p \geq 13$ ,  $p \equiv \pm 1 (5)$ ,  $p \not\equiv \pm 1 (8)$   
and  $d \geq 15$ ,
- (c)  $g = 1 + \frac{p(p^2-1)}{48}$  if  $p \geq 13$ ,  $p \not\equiv \pm 1 (5)$ ,  $p \equiv \pm 1 (8)$   
and  $d \geq 12$ ,
- (d)  $g = 1 + \frac{p(p^2-1)}{80}$  if  $p \geq 13$ ,  $p \equiv \pm 1 (5)$ ,  $p \equiv \pm 1 (8)$   
and  $d \geq 9$ ,
- (e)  $g = 1 + \frac{p(p^2-1)}{4} \left( \frac{1}{6} - \frac{1}{d} \right)$  in all other cases.

In fact the least genus  $g$  always comes from the branched covering space action on the Riemann surface  $S = H/\Delta$  associated to some extension  $1 \rightarrow \Delta \rightarrow T(r, s, t) \rightarrow PSl_2(p) \rightarrow 1$ , where

$$(r, s, t) = \begin{cases} (2, 3, p) & \text{if } p = 5, 7, 11, \\ (2, 5, 5) & \text{if } p \geq 13, p \equiv \pm 1 (5), p \not\equiv \pm 1 (8) \text{ and } d \geq 15, \\ (3, 3, 4) & \text{if } p \geq 13, p \not\equiv \pm 1 (5), p \equiv \pm 1 (8) \text{ and } d \geq 12, \\ (2, 4, 5) & \text{if } p \geq 13, p \equiv \pm 1 (5), p \equiv \pm 1 (8) \text{ and } d \geq 9, \\ (2, 3, d) & \text{in all other cases.} \end{cases}$$

It turns out that other triples  $(r, s, t)$  are not relevant for the determination of the minimal genus.

In most cases the answer is  $(r, s, t) = (2, 3, d)$ . For  $p \leq 617$  the triple  $(2, 5, 5)$  occurs once exactly, namely for  $p = 509$ ,  $(3, 3, 4)$  occurs exactly three

times, namely for  $p = 103, 137$  and  $569$  and  $(2, 4, 5)$  occurs exactly six times, for  $p = 199, 239, 359, 439, 521$  and  $599$ .

If  $S = H/\Delta$  is the surface of minimal genus for  $PSl_2(p)$  coming from one of the extensions above then the orbit manifold  $S/PSl_2(p)$  is the 2-sphere  $S^2$  and the quotient map  $S \rightarrow S^2$  is a branched covering with exactly 3 branch points. One of the most important steps in the proof of the main result of this paper is the converse, namely if  $S$  is a Riemann surface of least genus for the group  $G = PSl_2(p)$  then  $S/G = S^2$  and  $S \rightarrow S^2$  is a branched covering with exactly 3 branch points (see section 3). Note that a related notion of genus, "the Cayley genus of a group" has been studied by others, among them Tucker [T]. Earlier results can be found in Hurwitz [H] and Burnside [B].

The remainder of this paper is organized as follows. In section 2 we describe various ways of generating  $PSl_2(p)$  and then prove theorems I and II. Section 3 proves that if  $S$  is a Riemann surface of least genus for  $PSl_2(p)$  then  $S/PSl_2(p)$  is a 2-sphere  $S^2$  and the branched covering  $S \rightarrow S^2$  has exactly 3 branch points. The calculation of genus ( $PSl_2(p)$ ) then follows from the results of section 2.

Finally we would like to thank Bomshik Chang for help with the group theory of  $PSl_2(p)$ . The first author would like to thank the University of British Columbia for its hospitality to him during the time this research was done.

## § 2. GENERATING TRIPLES FOR $PSl_2(p)$

Our goal in this section is to find triples  $(r, s, t)$  for which there are epimorphisms  $T(r, s, t) \rightarrow PSl_2(p)$ . In other words, given integers  $r, s, t \geq 2$  are there matrices  $A, B, C \in PSl_2(p)$  so that  $A, B, C$  generate  $PSl_2(p)$  and  $A^r = B^s = C^t = ABC = 1$ ? Throughout this section a standard reference for the group theory is Suzuki [S].

The spherical triangle groups are given in the following table

TABLE I

<i>triple</i>	<i>triangle group</i>	<i>order</i>
$(2, 2, n)$	dihedral	$2n$
$(2, 3, 3)$	tetrahedral ( $A_4$ )	12
$(2, 3, 4)$	octahedral ( $S_4$ )	24
$(2, 3, 5)$	icosahedral ( $A_5$ )	60